# 1. Path Integral Representation of Quantum Mechanics

There is a Hamiltonian formalism and a Lagrangian formalism for quantum mechanics just as for analytical mechanics. The wave mechanics of Schrödinger and the matrix mechanics of Heisenberg belong to the Hamiltonian formalism, while the path integral quantization of Feynman which originates from Dirac belongs to the Lagrangian formalism.

We can deduce wave mechanics from the Hamilton–Jacobi equation of analytical mechanics. We regard the Hamilton–Jacobi equation as the Eikonal equation of "geometrical mechanics". We obtain the Eikonal equation of geometrical optics from wave optics by the Eikonal approximation. We apply the inverse of the Eikonal approximation to the Hamilton–Jacobi equation and obtain the governing equation of "wave mechanics", which is the Schrödinger equation. We can deduce likewise the "matrix mechanics" of Heisenberg from consideration of the consistency of the Ritz combination principle, the Bohr quantization condition and the Fourier analysis of physical quantities in classical physics.

We can deduce path integral quantization from consideration of the similarity of the contact transformation in analytical mechanics and quantum mechanics. We arrive at path integral quantization by regarding the time evolution of the quantum mechanical system as the convolution of an infinitesimal contact transformation. We express the transformation function $\langle q_\mathrm{f}, t_\mathrm{f} | q_\mathrm{i}, t_\mathrm{i} \rangle$ in terms of the action functional (the time integral of the Lagrangian) along all possible paths connecting the initial state and the final state. Path integral quantization is the integral form of the quantum mechanical action principle, embodying the principle of superposition and the composition law of the transition probability amplitudes. We can easily compare the quantum mechanical result with the classical result in this approach. In this chapter, we address ourselves to the path integral representation of quantum mechanics.

In Sect. 1.1, we discuss quantum mechanics in the Lagrangian formalism. We discuss the contact transformation in analytical mechanics and quantum mechanics (Sect. 1.1.1), the relationship with the action principle in analytical mechanics (Sect. 1.1.2), the trivial statement of the quantum mechanical action principle (Sect. 1.1.3), the derivation of the time-dependent Schrödinger

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equation (Sect. 1.1.4), and the principle of superposition and the composition law of the transition probability amplitude (Sect. 1.1.5).

In Sect. 1.2, we discuss quantum mechanics in the Hamiltonian formalism. We review quantum mechanics in the Hamiltonian formalism (Sect. 1.2.1). We have the configuration space path integral formula from the outset for quantum mechanics in the Lagrangian formalism. In the Hamiltonian formalism, however, the transformation function (Sect. 1.2.2), the matrix element of the time-ordered product (Sect. 1.2.3), and, by way of the wave function of the vacuum (Sect. 1.2.4), the generating functional of the Green's function (Sect. 1.2.5) are all given by the phase space path integral formula to begin with. When the Hamiltonian is given by the quadratic form of the canonical momentum, we can perform the momentum functional integration easily as a quasi-Gaussian integral. When the kernel of the quadratic part of the canonical momentum is the constant matrix, we recover the configuration space path integral formula of the Lagrangian formalism (Sect. 1.2.6) with the use of the canonical equations of motion. When the kernel is not the constant matrix but a $q$-dependent matrix, we have a $q$-dependent determinant factor in the functional integrand in the configuration space path integral formula, and recognize the fact that the Feynman path integral formula obtained in Sect. 1.1.3 is not always correct. In this case, by replacing the original Lagrangian of the mechanical system with the effective Lagrangian which takes the presence of the $q$-dependent determinant factor into consideration, we can use the Feynman path integral formula.

We evade the important problem of operator ordering in the transition from analytical mechanics to quantum mechanics with the use of the notion of a "well-ordered" operator (introduced in Sect. 1.1.1) throughout Sects. 1.1 and 1.2. In Sect. 1.3, we deal with the operator ordering problem squarely. We employ the Weyl correspondence as the prescription of the operator ordering problem, and we discuss the Weyl correspondence in Sect. 1.3.1. We reconsider the path integral formula in the Cartesian coordinate system (Sect. 1.3.2) and in the curvilinear coordinate system (Sect. 1.3.3) under the Weyl correspondence. We shall derive the mid-point rule as a natural consequence of the Weyl correspondence. A noteworthy point is the emergence of the new effective potential which originates from the normalization of the wave function in the curvilinear coordinate system and the Jacobian of the coordinate transformation. Generally speaking, we can use the Feynman path integral formula if we replace the original Lagrangian with the new effective Lagrangian which takes the $q$-dependent determinant factor (which also shows up in Sect. 1.3) and the new effective potential into consideration.

## 1.1 Quantum Mechanics in the Lagrangian Formalism

The wave mechanics and the matrix mechanics stated at the beginning of this chapter, which belong to the Hamiltonian formalism, are built on the analogy with the Hamiltonian formalism of analytical mechanics. Namely, the coordinate operator $\hat{q}(t)$, the canonically conjugate momentum operator $\hat{p}(t)$ and the equal time canonical commutator $[\hat{q}(t), \hat{p}(t)]$ of quantum mechanics correspond to the coordinate $q(t)$, the canonically conjugate momentum $p(t)$ and the Poisson bracket $\{q(t), p(t)\}_{\mathrm{PB}}$ of analytical mechanics in the Hamiltonian formalism. This correspondence makes the comparison of quantum mechanics with analytical mechanics easy. One good example will be the Ehrenfest theorem about the expectation value of the quantum-mechanical operator. On the other hand, it brings the difficulty of analytical mechanics in the Hamiltonian formalism directly into quantum mechanics in the Hamiltonian formalism. Good examples will be the problem of the normal dependence and the definition of the canonically conjugate momentum in the singular Lagrangian system to be discussed in Chaps. 2 and 3. Analytical mechanics in the Lagrangian formalism does not present such difficulties.

In a one-dimensional particle system, from the Lagrangian,

$$L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right),$$

which is a function of the coordinate $q(t)$ and the velocity $\mathrm{d}q(t)/\mathrm{d}t$, we construct the action functional,

$$I[q] \equiv \int \mathrm{d}t L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right),$$

which is a Lorentz invariant scalar. From Hamilton's action principle, we obtain the Euler–Lagrange equation of motion,

$$\frac{\delta I[q]}{\delta q(t)} = \frac{\partial L(q(t), \mathrm{d}q(t)/\mathrm{d}t)}{\partial q(t)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(q(t), \mathrm{d}q(t)/\mathrm{d}t)}{\partial(\mathrm{d}q(t)/\mathrm{d}t)}\right) = 0.$$

In this section, we examine the action principle in quantum mechanics in the Lagrangian formalism with the use of the contact transformation, following the classic papers by Dirac and Feynman.

### 1.1.1 Contact Transformation in Analytical Mechanics and Quantum Mechanics

We consider analytical mechanics in the Lagrangian formalism described by the Lagrangian $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)$ with $f$ degree of freedom, where $q_r(t)$

are the generalized coordinates and $\mathrm{d}q_r(t)/\mathrm{d}t$ are the generalized velocities $(r = 1, \ldots, f)$. The action functional $I[q; q(t_b), q(t_a)]$ is given by

$$I[q; q(t_b), q(t_a)] \equiv \int_{t_a}^{t_b} \mathrm{d}t L\left(q_r(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r(t)\right). \tag{1.1.1}$$

Hamilton's action principle demands that the action functional

$$I[q; q(t_b), q(t_a)]$$

takes a stationary value for an infinitesimal variation $\delta q_r(t)$ of $q_r(t)$ with the end-points, $q_r(t_a)$ and $q_r(t_b)$, fixed $(r = 1, \ldots, f)$:

$$\delta I[q; q(t_b), q(t_a)] = 0; \qquad \delta q_r(t_a) = \delta q_r(t_b) = 0, \quad r = 1, \ldots, f. \tag{1.1.2}$$

From (1.1.2), we immediately obtain the Euler–Lagrange equation of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)}{\partial(\mathrm{d}q_r(t)/\mathrm{d}t)}\right) - \frac{\partial L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)}{\partial q_r(t)} = 0, \quad r = 1, \ldots, f. \tag{1.1.3}$$

We move on to the Hamiltonian formalism of analytical mechanics by adopting the following procedure. We first define the momentum $p_r(t)$, $(r = 1, \ldots, f)$, canonically conjugate to the generalized coordinate $q_r(t)$, $(r = 1, \ldots, f)$, by the following equation:

$$p_r(t) \equiv \frac{\partial L(q_s(t), \mathrm{d}q_s(t)/\mathrm{d}t)}{\partial\left(\mathrm{d}q_r(t)/\mathrm{d}t\right)}, \quad r, s = 1, \ldots, f. \tag{1.1.4}$$

We solve (1.1.4) for $\mathrm{d}q_r(t)/\mathrm{d}t$ as a function of $q_s(t)$ and $p_s(t)$, $(s = 1, \ldots, f)$. Next, we define the Hamiltonian $H(q_r(t), p_r(t))$ as the Legendre transform of the Lagrangian $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)$ by the following equation,

$$H(q_r(t), p_r(t)) \equiv \sum_{s=1}^{f} p_s(t) \frac{\mathrm{d}}{\mathrm{d}t} q_s(t) - L\left(q_r(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r(t)\right), \tag{1.1.5}$$

where we substitute $\mathrm{d}q_r(t)/\mathrm{d}t$, expressed as a function of $q_s(t)$ and $p_s(t)$, $(r, s = 1, \ldots, f)$, into the right-hand side of (1.1.5). Finally, we take the independent variations of $q_r(t)$ and $p_r(t)$ in (1.1.5). Making use of (1.1.3) and (1.1.4), we obtain Hamilton's canonical equations of motion:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q_r(t) &= \frac{\partial H(q_s(t), p_s(t))}{\partial p_r(t)}, \\ \frac{\mathrm{d}}{\mathrm{d}t} p_r(t) &= -\frac{\partial H(q_s(t), p_s(t))}{\partial q_r(t)}, \qquad r, s = 1, \ldots, f. \end{aligned} \tag{1.1.6}$$

The first equation of (1.1.6) declares that $\mathrm{d}q_r(t)/\mathrm{d}t$ is a function of $q_s(t)$ and

$p_s(t)$, $(s = 1, \ldots, f)$, explicitly. Conversely, we can obtain the Euler–Lagrange equation of motion, (1.1.3), and the definition of the momentum $p_r(t)$ canonically conjugate to the coordinate $q_r(t)$, (1.1.4), from the definition of the Hamiltonian, (1.1.5), and Hamilton's canonical equation of motion, (1.1.6). As long as we can invert the definition of the canonical momentum $p_r(t)$, (1.1.4), for $\mathrm{d}q_r(t)/\mathrm{d}t$, or the Lagrangian $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)$ is nonsingular, we have the equivalence of the Euler–Lagrange equation of motion, (1.1.3), and Hamilton's canonical equations of motion, (1.1.6).

We move on to a discussion of canonical transformation theory. We rewrite the definition of the Hamiltonian $H(q_r(t), p_r(t))$, (1.1.5), as follows:

$$L\left(q_r(t), \frac{\mathrm{d}}{\mathrm{d}t}q_r(t)\right) = \sum_{s=1}^{f} p_s(t)\frac{\mathrm{d}}{\mathrm{d}t}q_s(t) - H(q_r(t), p_r(t))\,. \tag{1.1.7}$$

We recall that $\mathrm{d}q_r(t)/\mathrm{d}t$ in (1.1.7) is regarded as a function of $q_s(t)$ and $p_s(t)$, $(s = 1, \ldots, f)$, and that the left-hand side of (1.1.7), $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)$, is not a Lagrangian, but a function of $q_s(t)$ and $p_s(t)$, $(s = 1, \ldots, f)$, defined by the right-hand side of (1.1.7). We remark that the variation of $q_r(t)$,

$$q_r(t) \to q_r(t) + \delta q_r(t),$$

used in the extremization of the action functional $I[q; q(t_b), q(t_a)]$ induces the variation of $\mathrm{d}q_r(t)/\mathrm{d}t$,

$$\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) \to \frac{\mathrm{d}}{\mathrm{d}t}q_r(t) + \delta\left(\frac{\mathrm{d}}{\mathrm{d}t}q_r(t)\right),$$

which in turn induces the variation of $p_r(t)$,

$$p_r(t) \to p_r(t) + \delta p_r(t),$$

since $\mathrm{d}q_r(t)/\mathrm{d}t$ is a function of $q_s(t)$ and $p_s(t)$, $(r, s = 1, \ldots, f)$. This is the meaning of the independent variation of $q_r(t)$ and $p_r(t)$, used to derive Hamilton's canonical equations of motion (1.1.6). We now consider the transformation of the pair of canonical variables from $(q_r(t), p_r(t))$, $(r = 1, \ldots, f)$, to $(Q_r(t), P_r(t))$, $(r = 1, \ldots, f)$. This transformation,

$$(q_r(t), p_r(t)) \to (Q_r(t), P_r(t)), \quad r = 1, \ldots, f,$$

is said to be a canonical transformation if the following condition holds:

$$\begin{aligned}\delta\int_{t_a}^{t_b} dt &\left(\sum_{s=1}^{f} p_s(t)\frac{\mathrm{d}}{\mathrm{d}t}q_s(t) - H(q_r(t), p_r(t))\right) \\ &= \delta\int_{t_a}^{t_b} \mathrm{d}t\left(\sum_{s=1}^{f} P_s(t)\frac{\mathrm{d}}{\mathrm{d}t}Q_s(t) - \bar{H}(Q_r(t), P_r(t))\right) \\ &= 0\,,\end{aligned} \tag{1.1.8}$$

with

$$\delta q_r(t_{a,b}) = \delta p_r(t_{a,b}) = \delta Q_r(t_{a,b}) = \delta P_r(t_{a,b}) = 0, \quad r = 1, \ldots, f.$$

The new canonical pair $(Q_r(t), P_r(t))$, $(r = 1, \ldots, f)$, satisfies Hamilton's canonical equation of motion, (1.1.6), with $\bar{H}(Q_r(t), P_r(t))$ as the new Hamiltonian. Since we have

$$\delta q_r(t_{a,,b}) = \delta p_r(t_{a,b}) = \delta Q_r(t_{a,b}) = \delta P_r(t_{a,b}) = 0, \quad r = 1, \ldots, f,$$

(1.1.8) is equivalent to the following equation:

$$\sum_{s=1}^{f} p_s(t)\frac{\mathrm{d}}{\mathrm{d}t}q_s(t) - H(q_r(t), p_r(t)) = \sum_{s=1}^{f} P_s(t)\frac{\mathrm{d}}{\mathrm{d}t}Q_s(t) - \bar{H}(Q_r(t), P_r(t)) + \frac{\mathrm{d}}{\mathrm{d}t}U\,. \tag{1.1.9}$$

In (1.1.9), $U$ is a function of an arbitrary pair of $\{q_r(t), p_r(t), Q_r(t), P_r(t)\}$, $(r = 1, \ldots, f)$ and $t$, and is a single-valued continuous function. We call $U$ the generator of the canonical transformation. When we choose $U$ to be the function $S(q_r(t), Q_r(t))$ of $q_r(t)$ and $Q_r(t)$, $(r = 1, \ldots, f)$,

$$U = S(q_r(t), Q_r(t))\,, \tag{1.1.10}$$

which does not depend on $t$ explicitly, we call the canonical transformation generated by $U$ a contact transformation. Since we have, from (1.1.10),

$$\frac{\mathrm{d}}{\mathrm{d}t}U = \sum_{s=1}^{f}\left\{\frac{\partial S(q_r(t), Q_r(t))}{\partial q_s(t)}\frac{\mathrm{d}}{\mathrm{d}t}q_s(t) + \frac{\partial S(q_r(t), Q_r(t))}{\partial Q_s(t)}\frac{\mathrm{d}}{\mathrm{d}t}Q_s(t)\right\}, \tag{1.1.11}$$

we obtain, from (1.1.9), the contact transformation formula in analytical mechanics:

$$p_r(t) = \frac{\partial S(q_s(t), Q_s(t))}{\partial q_r(t)},$$

$$P_r(t) = -\frac{\partial S(q_s(t), Q_s(t))}{\partial Q_r(t)}, \qquad r = 1, \ldots, f\,, \tag{1.1.12a}$$

$$H(q_r(t), p_r(t)) = \bar{H}(Q_r(t), P_r(t))\,. \tag{1.1.12b}$$

We now seek a quantum analog of the contact transformation formula, (1.1.12a) and (1.1.12b), after Dirac. We choose the eigenkets $|q\rangle$ and $|Q\rangle$ which make the operators $\hat{q}_r$ and $\hat{Q}_r$ diagonal:

$$\hat{q}_r\,|q\rangle = q_r\,|q\rangle\,, \qquad \hat{Q}_r\,|Q\rangle = Q_r\,|Q\rangle\,, \qquad r = 1, \ldots, f\,. \tag{1.1.13}$$

We state the conclusions first. We have the following quantum-mechanical contact transformation formula:

$$\langle q|\, Q\rangle \;= \exp\left[\frac{\mathrm{i}}{\hbar}G(q,Q)\right], \tag{1.1.14}$$

$$G(q,Q) = \text{quantum analog of } S(q,Q)\,, \tag{1.1.15}$$

$$\begin{aligned}\langle q|\, Q\rangle \;&= \exp\left[\frac{\mathrm{i}}{\hbar}G(q,Q)\right] \\ &= \text{quantum analog of } \exp\left[\frac{\mathrm{i}}{\hbar}S(q,Q)\right],\end{aligned} \tag{1.1.16}$$

$$\hat{p}_r = \frac{\partial \widehat{G(q,Q)}}{\partial q_r}, \qquad \hat{P}_r = -\frac{\partial \widehat{G(q,Q)}}{\partial Q_r}, \qquad r = 1,\dots,f\,. \tag{1.1.17}$$

We will begin with the mixed representative $\langle q|\,\hat{\alpha}\,|Q\rangle$ of the dynamical variable $\hat{\alpha}$:

$$\langle q|\hat{\alpha}|Q\rangle = \int \langle q|\hat{\alpha}|q'\rangle \mathrm{d}q' \langle q'|Q\rangle \tag{1.1.18a}$$

$$= \int \langle q|Q'\rangle \mathrm{d}Q' \langle Q'|\hat{\alpha}|Q\rangle\,. \tag{1.1.18b}$$

In (1.1.18a) and (1.1.18b), we let $\hat{\alpha}$ be $\hat{q}_r$, $\hat{p}_r$, $\hat{Q}_r$ and $\hat{P}_r$, obtaining

$$\langle q|\,\hat{q}_r\,|Q\rangle \;= q_r\;\langle q|\,Q\rangle, \qquad \langle q|\,\hat{p}_r\,|Q\rangle \;= \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q_r}\;\langle q|\,Q\rangle\,, \tag{1.1.19}$$

$$\langle q|\,\hat{Q}_r\,|Q\rangle \;= Q_r\;\langle q|\,Q\rangle, \qquad \langle q|\,\hat{P}_r\,|Q\rangle \;= -\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial Q_r}\;\langle q|\,Q\rangle\,. \tag{1.1.20}$$

Next, we introduce the notion of a *well-ordered* operator. When we say that $\alpha(\hat{q},\hat{Q})$ is a well-ordered operator, we mean that $\alpha(\hat{q},\hat{Q})$ can be written in the form

$$\alpha(\hat{q},\hat{Q}) = \sum_k f_k(\hat{q})g_k(\hat{Q})\,. \tag{1.1.21}$$

Then, from (1.1.19) and (1.1.20), we obtain the identity

$$\langle q|\,\alpha(\hat{q},\hat{Q})\,|Q\rangle \;= \alpha(q,Q)\;\langle q|\,Q\rangle\,. \tag{1.1.22}$$

We recall that, in (1.1.22), while $\alpha(\hat{q}, \hat{Q})$ on the left-hand side is a $q$-number function, $\alpha(q, Q)$ on the right-hand side is an ordinary $c$-number function. With these preparations, we write the transformation function $\langle q|Q\rangle$ in the form (1.1.14), and apply the second equations of (1.1.19) and (1.1.20). Then we obtain

$$\begin{aligned}\langle q|\hat{p}_r|Q\rangle &= \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q_r}\langle q|Q\rangle = \frac{\partial G(q,Q)}{\partial q_r}\langle q|Q\rangle \\ &= \left\langle q \left| \frac{\partial \widehat{G(q,Q)}}{\partial q_r} \right| Q \right\rangle , \end{aligned} \tag{1.1.23}$$

$$\begin{aligned}\langle q|\hat{P}_r|Q\rangle &= -\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial Q_r}\langle q|Q\rangle = -\frac{\partial G(q,Q)}{\partial Q_r}\langle q|Q\rangle \\ &= -\left\langle q \left| \frac{\partial \widehat{G(q,Q)}}{\partial Q_r} \right| Q \right\rangle . \end{aligned} \tag{1.1.24}$$

In (1.1.23) and (1.1.24), the last equalities hold in the sense of a well-ordered operator. Thus, we have (1.1.17) in the sense of a well-ordered operator identity. From a comparison of (1.1.17) with (1.1.12), we obtain (1.1.15) and (1.1.16). Equations (1.1.14) through (1.1.17) are the contact transformation formula in quantum mechanics. We cannot drop the “quantum analog” in (1.1.15) and (1.1.16), since $G(q, Q)$ is a complex number in general.

### 1.1.2 The Lagrangian and the Action Principle

In this section, we consider the description of the time evolution of an analytical mechanical system and a quantum mechanical system in terms of the contact transformation, and the relationship with the action principle.

In the discussion so far, we regarded the action functional $I[q; q(t_b), q(t_a)]$ as the time integral of the Lagrangian $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t))$ along the arbitrary path $\{q_r(t); t_b \geq t \geq t_a\}_{r=1}^{f}$ connecting the fixed end-points, $q_r(t_a)$ and $q_r(t_b)$:

$$I[q; q(t_b), q(t_a)] = \int_{t_a}^{t_b} \mathrm{d}t L\left(q_r(t), \frac{\mathrm{d}}{\mathrm{d}t}q_r(t)\right) .$$

From the extremization of the action functional $I[q; q(t_b), q(t_a)]$ with the end-points fixed,

$$\delta q_r(t_a) = \delta q_r(t_b) = 0, \qquad r = 1, \ldots , f,$$

we obtained the Euler–Lagrange equation of motion, (1.1.3), and we have determined the classical path $\{q_r^{\mathrm{cl}}(t); t_b \geq t \geq t_a\}_{r=1}^{f}$. In this subsection, we consider the notion of the action functional from a different point of view. Namely, we consider the action functional as the time integral of the Lagrangian $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t))$ along the true classical path,

$$\left\{q_r^{\mathrm{cl}}(t); t_b \geq t \geq t_a\right\}_{r=1}^{f},$$

connecting the initial position, $q_r(t_a)$, and the final position, $q_r(t_b)$, $(r = 1, \ldots, f)$:

$$I(q_r(t_b), q_r(t_a)) \equiv \int_{t_a}^{t_b} \mathrm{d}t L\left(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t)\right). \tag{1.1.25}$$

In other words, we consider the action functional as a function of the end-points, $q_r(t_a)$ and $q_r(t_b)$, connected by the true classical path,

$$\left\{q_r^{\mathrm{cl}}(t); t_b \geq t \geq t_a\right\}_{r=1}^{f}.$$

We consider the infinitesimal variation of the end-points,

$$q_r(t_{a,b}) \to q_r(t_{a,b}) + \delta q_r(t_{a,b}).$$

We obtain the response of the action functional to this variation as

$$\begin{aligned}
&\delta I(q_r(t_b), q_r(t_a)) \\
&\quad = \int_{t_a}^{t_b} \mathrm{d}t \sum_{s=1}^{f} \left\{ \frac{\partial L(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t))}{\partial q_s^{\mathrm{cl}}(t)} \delta q_s^{\mathrm{cl}}(t) \right. \\
&\qquad \left. + \frac{\partial L(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t))}{\partial(\frac{\mathrm{d}}{\mathrm{d}t} q_s^{\mathrm{cl}}(t))} \delta\left(\frac{\mathrm{d}}{\mathrm{d}t} q_s^{\mathrm{cl}}(t)\right) \right\} \\
&\quad = \sum_{s=1}^{f} \int_{t_a}^{t_b} \mathrm{d}t \left[ \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t))}{\partial\left(\frac{\mathrm{d}}{\mathrm{d}t} q_s^{\mathrm{cl}}(t)\right)} \delta q_s^{\mathrm{cl}}(t) \right) \right. \\
&\qquad + \left\{ \frac{\partial L(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t))}{\partial q_s^{\mathrm{cl}}(t)} \right. \\
&\qquad \left. \left. - \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t))}{\partial\left(\frac{\mathrm{d}}{\mathrm{d}t} q_s^{\mathrm{cl}}(t)\right)} \right) \right\} \delta q_s^{\mathrm{cl}}(t) \right].
\end{aligned} \tag{1.1.26}$$

By the definition of $I(q_r(t_b), q_r(t_a))$, (1.1.25), the second term in the last equality in (1.1.26) vanishes due to the Euler–Lagrange equation of motion, (1.1.3). Using the definition of the momentum $p_r(t)$ canonically conjugate to $q_r(t)$, (1.1.4), we obtain

$$\delta I(q_r(t_b), q_r(t_a)) = \sum_{s=1}^{f} [p_s(t_b)\delta q_s(t_b) - p_s(t_a)\delta q_s(t_a)]. \tag{1.1.27}$$

From (1.1.27), we obtain

$$p_r(t_b) = \frac{\partial I(q_s(t_b), q_s(t_a))}{\partial q_r(t_b)},$$
$$p_r(t_a) = -\frac{\partial I(q_s(t_b), q_s(t_a))}{\partial q_r(t_a)}, \qquad r = 1, \ldots, f\,. \tag{1.1.28}$$

These correspond to the contact transformation formula in analytical mechanics, (1.1.12a), in Sect. 1.1.1, with the identification

$$(q_r, p_r) = (q_r(t_b), p_r(t_b)),$$
$$(Q_r, P_r) = (q_r(t_a), p_r(t_a)), \quad r = 1, \ldots, f\,, \tag{1.1.29}$$

$$S(q, Q) = I(q_r(t_b), q_r(t_a)) = \int_{t_a}^{t_b} \mathrm{d}t L\left(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t)\right). \tag{1.1.30}$$

We find that the time evolution of an analytical mechanical system,

$$(q_r(t_a), p_r(t_a)) \rightarrow (q_r(t_b), p_r(t_b)),$$

is given by the contact transformation, (1.1.28), with the generator

$$I(q_r(t_b), q_r(t_a)).$$

We now turn to the discussion of the transformation function $\langle q_{t_b} | q_{t_a} \rangle$ which generates the time evolution of a quantum mechanical system. We write $\langle q_{t_b} | q_{t_a} \rangle$ as,

$$\langle q_{t_b} | q_{t_a} \rangle = \exp\left[\frac{\mathrm{i}}{\hbar} G(q_{t_b}, q_{t_a})\right], \tag{1.1.31}$$

mimicking (1.1.14). Here, we have used the following shorthand

$$q_{t_a} = \{q_r(t_a)\}_{r=1}^{f}, \qquad q_{t_b} = \{q_r(t_b)\}_{r=1}^{f}.$$

Based on the quantum analog (1.1.15) and (1.1.16) deduced in Sect. 1.1.1, we conclude the following analog,

$$\begin{aligned} G(q_{t_b}, q_{t_a}) &= \text{quantum analog of } I(q_r(t_b), q_r(t_a)) \\ &= \text{quantum analog of } \int_{t_a}^{t_b} \mathrm{d}t L\left(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t)\right), \end{aligned} \tag{1.1.32}$$

$$\begin{aligned} \langle q_{t_b} | q_{t_a} \rangle &= \exp\left[\frac{\mathrm{i}}{\hbar} G(q_{t_b}, q_{t_a})\right] \\ &= \text{quantum analog of } \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t_a}^{t_b} \mathrm{d}t L\left(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r^{\mathrm{cl}}(t)\right)\right], \end{aligned} \tag{1.1.33}$$

$$\begin{aligned}\langle q_{t+\delta t}|q_t\rangle &= \exp\left[\frac{\mathrm{i}}{\hbar}G(q_{t+\delta t}, q_t)\right] \\ &= \text{quantum analog of } \exp\left[\frac{\mathrm{i}}{\hbar}\cdot\delta t\cdot L\left(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t}q_r^{\mathrm{cl}}(t)\right)\right]. \quad (1.1.34)\end{aligned}$$

From (1.1.34), we learn that it is better not to regard the Lagrangian $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)$ as a function of $q_r(t)$ and $\mathrm{d}q_r(t)/\mathrm{d}t$ but rather as a function of $q_r(t)$ and $q_r(t+\delta t)$, i.e., the Lagrangian $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)$ is a dipole quantity. We can rewrite (1.1.34) as

$$\langle q_{t+\delta t}|q_t\rangle = \text{quantum analog of } \exp\left[\frac{\mathrm{i}}{\hbar}\cdot\delta t\cdot L(q_r(t), \text{“}q_r(t+\delta t)\text{”})\right] \quad (1.1.35)$$

where “$q_r(t+\delta t)$” indicates that we approximate $\mathrm{d}q_r(t)/\mathrm{d}t$ by

$$\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) \simeq \frac{q_r(t+\delta t) - q_r(t)}{\delta t} \quad (1.1.36)$$

to the first order in $\delta t$. We have clarified the relationship between the time evolution of an analytical mechanical system and that of quantum mechanical system at the level of the quantum analog with the discussion of the contact transformation generated by the action functional, $I(q_r(t_b), q_r(t_a))$.

Next, we discuss the relationship between the action principle in analytical mechanics and quantum mechanics with the use of (1.1.33) at the level of the quantum analog. We introduce the new notation $B(t_b, t_a)$, which is a classical quantity and corresponds to the transformation function $\langle q_{t_b}|q_{t_a}\rangle$ in quantum mechanics:

$$\begin{aligned}B(t_b, t_a) &\equiv \exp\left[\frac{\mathrm{i}}{\hbar}I(q_r(t_b), q_r(t_a))\right] \\ &= \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}\mathrm{d}tL\left(q_r^{\mathrm{cl}}(t), \frac{\mathrm{d}}{\mathrm{d}t}q_r^{\mathrm{cl}}(t)\right)\right]. \quad (1.1.37)\end{aligned}$$

We divide the time interval $t_b \geq t \geq t_a$ into $n$ equal subintervals

$$t_k = t_a + \frac{k}{n}(t_b - t_a), \quad 0 \leq k \leq n; \quad t_0 = t_a, \quad t_n = t_b,$$

and designate the values of $q_r(t)$ at $t = t_k$ by

$$q_{t_k} \equiv \{q_r(t_k)\}_{r=1}^{f}.$$

We have the following identity for the classical quantity $B(t_b, t_a)$:

$$\begin{aligned}B(t_b, t_a) &= B(t_b, t_{n-1})B(t_{n-1}, t_{n-2}) \\ &\quad \cdots B(t_k, t_{k-1})\cdots B(t_2, t_1)B(t_1, t_a)\,. \quad (1.1.38)\end{aligned}$$

The right-hand side of (1.1.38) is generally a function of $q_{t_b}$, $q_{t_{n-1}}$, $\ldots$,$q_{t_1}$, $q_{t_a}$. For each $q_{t_k}$, the values of the classical path $\{q_r^{\rm cl}(t_k)\}_{r=1}^{f}$ determined by the action principle are substituted. As a result of the additivity of the action functional, $I(q_r(t_k), q_r(t_{k-1}))$, we end up with a function of $q_{t_b}$ and $q_{t_a}$ alone as the left-hand side of (1.1.38) indicates. On the other hand, in quantum mechanics we have the following composition law for the transformation function $\langle q_{t_b}|q_{t_a}\rangle$ with repeated use of resolution of the identity,

$$\langle q_{t_b}|q_{t_a}\rangle = \int\cdots\int \langle q_{t_b}|q_{t_{n-1}}\rangle {\rm d}q_{t_{n-1}}\langle q_{t_{n-1}}|q_{t_{n-2}}\rangle {\rm d}q_{t_{n-2}}\langle q_{t_{n-2}}| \\ \cdots |q_{t_2}\rangle {\rm d}q_{t_2}\langle q_{t_2}|q_{t_1}\rangle {\rm d}q_{t_1}\langle q_{t_1}|q_{t_a}\rangle\,. \tag{1.1.39}$$

We apply the previously deduced analog between $\langle q_{t_b}|q_{t_a}\rangle$ and $B(t_b, t_a)$, (1.1.33), to the right-hand side of (1.1.39). We conclude that the integrand on the right-hand side of (1.1.39) must be of the form

$$\text{integrand of (1.1.39)} = \exp\left[\frac{\rm i}{\hbar}F(q_{t_b}, q_{t_{n-1}}, \ldots, q_{t_1}, q_{t_a})\right]. \tag{1.1.40}$$

Here, the function in the exponent of the right-hand side of (1.1.40), $F(\cdots)$, is a finite function in the limit, $\hbar \to 0$. From (1.1.33), we obtain the following quantum analog,

$$\begin{aligned} &F(q_{t_b}, q_{t_{n-1}}, \ldots, q_{t_1}, q_{t_a}) \\ &\quad= \text{quantum analog of } \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} {\rm d}tL\left(q_r^{\rm cl}(t), \frac{\rm d}{{\rm d}t}q_r^{\rm cl}(t)\right) \\ &\quad= \text{quantum analog of } \int_{t_a}^{t_b} {\rm d}tL\left(q_r^{\rm cl}(t), \frac{\rm d}{{\rm d}t}q_r^{\rm cl}(t)\right) \\ &\quad= \text{quantum analog of } I(q_r(t_b), q_r(t_a))\,. \end{aligned} \tag{1.1.41}$$

We get the dominant contribution to the right-hand side of (1.1.39) from the neighborhood of

$$\{q_{t_k}^*, k = 1, \ldots, n-1\}$$

which extremizes the function

$$F(q_{t_b}, q_{t_{n-1}}, \ldots, q_{t_1}, q_{t_a})$$

against all infinitesimal variations $\delta q_{t_k}$ of $q_{t_k}$. According to (1.1.41), we have the extremized action $I(q_r(t_b), q_r(t_a))$ as the classical limit ($\hbar \to 0$) of $F(q_{t_b}, q_{t_{n-1}}, \ldots, q_{t_1}, q_{t_a})$.

Thus, we obtain the two trivial statements of the quantum-mechanical action principle:

(1) $\int \prod_{k=1}^{n-1} \mathrm{d}q_{t_k}$ of (1.1.39) which gives the transformation function $\langle q_{t_b} | q_{t_a} \rangle$ in quantum mechanics corresponds to the substitution of the classical path

$$\{q_r^{\mathrm{cl}}(t_k)\}_{r=1}^{f}$$

determined by the action principle at each time $t = t_k$ into the classical quantity $B(t_b, t_a)$.

(2) The extremization of $F(q_{t_b}, q_{t_{n-1}}, \dots, q_{t_1}, q_{t_a})$ of quantum mechanics corresponds to the extremization of the action functional $I[q_r; q_{t_b}, q_{t_a}]$ of analytical mechanics. In the classical limit ($\hbar \to 0$), the quantum analog of the action principle agrees with the action principle of analytical mechanics.

### 1.1.3 The Feynman Path Integral Formula

In the transformation function $\langle q_{t+\delta t} | q_t \rangle$, (1.1.35), which gives the time evolution of the quantum-mechanical system for the infinitesimal time interval, $[t, t+\delta t]$, Feynman replaced the quantum analog with equality, to the first order in $\delta t$, by multiplying by a constant factor $A$ on the right-hand side of (1.1.35):

$$\langle q_{t+\delta t} | q_t \rangle = A \cdot \exp\left[ \frac{\mathrm{i}}{\hbar} \cdot \delta t \cdot L(q_r(t),\ \text{“}q_r(t+\delta t)\text{”}) \right]. \tag{1.1.42}$$

In (1.1.39) which gives the transformation function $\langle q_{t_b} | q_{t_a} \rangle$ for the finite time interval $[t_a, t_b]$, we choose $\delta t$ and $t_k$ as follows:

$$\delta t = \frac{t_b - t_a}{n}, \quad t_k = t_a + k \cdot \delta t, \quad k = 0, 1, \dots, n-1, n;$$
$$t_0 = t_a, \quad t_n = t_b\,. \tag{1.1.43}$$

We then obtain the Feynman path integral representation for the transformation function $\langle q_{t_b} | q_{t_a} \rangle$ from (1.1.42):

$$\begin{aligned}
\langle q_{t_b} | q_{t_a} \rangle &= \int_{q_{t_0} = q_{t_a}}^{q_{t_n} = q_{t_b}} \prod_{k=1}^{n-1} \mathrm{d}q_{t_k} \prod_{k=0}^{n-1} \langle q_{t_{k+1}} | \, q_{t_k} \rangle \\
&= \lim_{\substack{n\to\infty \\ n\cdot\delta t = t_b - t_a}} \int_{q_{t_0} = q_{t_a}}^{q_{t_n} = q_{t_b}} \prod_{k=1}^{n-1} \mathrm{d}q_{t_k} \cdot \prod_{k=0}^{n-1} \langle q_{t_k + \delta t} | q_{t_k} \rangle \\
&= \lim_{\substack{n\to\infty \\ n\cdot\delta t = t_b - t_a}} A^n \int_{q_{t_0} = q_{t_a}}^{q_{t_n} = q_{t_b}} \prod_{k=1}^{n-1} \mathrm{d}q_{t_k} \\
&\quad \cdot \exp\left[ \frac{\mathrm{i}}{\hbar} \sum_{k=0}^{n-1} \delta t \cdot L(q_r(t_k),\ \text{“}q_r(t_k + \delta t)\text{”}) \right]
\end{aligned} \tag{1.1.44}$$

$$\equiv \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t L\left(q_r(t), \frac{\mathrm{d}}{\mathrm{d}t}q_r(t)\right)\right] \tag{1.1.45}$$
$$= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \exp\left[\frac{\mathrm{i}}{\hbar}I[q_r;q_{t_b},q_{t_a}]\right].$$

Here, we note that (1.1.45) is nothing but an elegant way of writing (1.1.44). We have the Riemann sum in (1.1.44) which is a sum over $k$ of the Lagrangian $L(q_r(t_k),$ "$q_r(t_{k+1})$") along the classical path

$$\left\{q_r^{\mathrm{cl}}(t), t_{k+1} \geq t \geq t_k\right\}_{r=1}^{f},$$

connecting $q_r(t_k)$ and $q_r(t_{k+1})$, and then we integrate over all $q_{t_k}$. In the limit as $n \to \infty$, this Riemann sum becomes the time integral of the Lagrangian $L(q_r(t), \mathrm{d}q_r(t)/\mathrm{d}t)$ along an arbitrary path,

$$\{q_r(t), t_b \geq t \geq t_a\}_{r=1}^{f},$$

connecting $q_r(t_a)$ and $q_r(t_b)$, and hence it is the action functional $I[q_r;q_{t_b},q_{t_a}]$ in the sense of (1.1.1). For this reason, we have dropped the superscript "cl" in (1.1.44) and (1.1.45). From this observation, we have the following statement,

$$\int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)]$$
$$= \text{the sum along all possible paths, } \{q_r(t), t_b \geq t \geq t_a\}_{r=1}^{f}, \text{ connecting the fixed end-points, } q_r(t_a) \text{ and } q_r(t_b). \tag{1.1.46}$$

Next, we consider the matrix elements $\langle q_{t_b}|O(\hat{q}(t))|q_{t_a}\rangle$ of the operator $O(\hat{q}(t))$, $(t_b \geq t \geq t_a)$. For this purpose, we need the principle of superposition and the composition law of the transition probability amplitude stated in the following form

$$\int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] = \int_{-\infty}^{+\infty} \mathrm{d}q' \int_{q_{\mathrm{II}}(t)=q'}^{q_{\mathrm{II}}(t_b)=q_{t_b}} \mathcal{D}[q_{\mathrm{II}}(t)] \int_{q_{\mathrm{I}}(t_a)=q_{t_a}}^{q_{\mathrm{I}}(t)=q'} \mathcal{D}[q_{\mathrm{I}}(t)]\,. \tag{1.1.47}$$

We let the eigenvalue and the eigenbra of the operator $\hat{q}(t)$ be given by $q'$ and $\langle q', t|$, respectively.

$$\langle q', t|\hat{q}(t) = \langle q', t|q'\,. \tag{1.1.48}$$

Applying the resolution of identity at time $t$,

$$1 = \int_{-\infty}^{+\infty} |q', t\rangle \mathrm{d}q' \langle q', t|\,, \tag{1.1.49}$$

to the matrix element $\langle q_{t_b}|O(\hat{q}(t))|q_{t_a}\rangle$, and making use of (1.1.45), (1.1.47) and (1.1.48), and the additivity of the action functional, we obtain

$$\begin{aligned}
\langle q_{t_b}|O(\hat{q}(t))|q_{t_a}\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}q' \langle q_{t_b}|q',t\rangle O(q')\langle q',t|q_{t_a}\rangle \\
&= \int_{-\infty}^{+\infty} \mathrm{d}q' \int_{q_{\mathrm{II}}(t)=q'}^{q_{\mathrm{II}}(t_b)=q_{t_b}} \mathcal{D}[q_{\mathrm{II}}(t)] \\
&\quad \times \exp\left[\frac{\mathrm{i}}{\hbar} I[q_{\mathrm{II}};q_{t_b},q']\right] O(q') \\
&\quad \times \int_{q_{\mathrm{I}}(t_a)=q_{t_a}}^{q_{\mathrm{I}}(t)=q'} \mathcal{D}[q_{\mathrm{I}}(t)] \exp\left[\frac{\mathrm{i}}{\hbar} I[q_{\mathrm{I}};q',q_{t_a}]\right] \\
&= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] O(q(t)) \exp\left[\frac{\mathrm{i}}{\hbar} I[q;q_{t_b},q_{t_a}]\right]. \quad (1.1.50)
\end{aligned}$$

As for the matrix element of the time-ordered product of $\hat{q}(t)$, we obtain, with repeated use of (1.1.47) and (1.1.50),

$$\begin{aligned}
&\langle q_{t_b}|\mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n))|q_{t_a}\rangle \\
&\quad = \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] q_{r_1}(t_1)\cdot\cdot q_{r_n}(t_n) \exp\left[\frac{\mathrm{i}}{\hbar} I[q_r;q_{t_b},q_{t_a}]\right]. \quad (1.1.51)
\end{aligned}$$

As for the generating functional $Z[J;q(t_b),q(t_a)]$ (and $W[J;q(t_b),q(t_a)]$) of the matrix element of the time-ordered product (and its connected part), we have

$$\begin{aligned}
Z[J;q(t_b),q(t_a)] &\equiv \exp\left[\frac{\mathrm{i}}{\hbar} W[J;q(t_b),q(t_a)]\right] \\
&\equiv \left\langle q_{t_b} \middle| \mathrm{T}\left(\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t \sum_{r=1}^{f} \hat{q}_r(t) J_r(t)\right]\right) \middle| q_{t_a} \right\rangle \\
&\equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\mathrm{i}}{\hbar}\right)^n \int_{t_a}^{t_b} \mathrm{d}t_1 \cdots \mathrm{d}t_n \\
&\quad \times \sum_{r_1,\ldots,r_n=1}^{f} J_{r_1}(t_1)\cdots J_{r_n}(t_n) \\
&\quad \times \langle q_{t_b}|\mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n))|q_{t_a}\rangle \\
&= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t \right. \\
&\quad \left. \times \left\{ L\left(q_r(t), \frac{\mathrm{d}}{\mathrm{d}t} q_r(t)\right) + \sum_{r=1}^{f} q_r(t) J_r(t) \right\}\right]. \quad (1.1.52)
\end{aligned}$$

The last line of (1.1.52) follows from (1.1.51). We have the formula

$$\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_1}(t_1)}\cdots\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_n}(t_n)}Z[J;q(t_b),q(t_a)]|_{J=0}$$
$$= \langle q_{t_b}|\mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n))|q_{t_a}\rangle\,, \tag{1.1.53Z}$$

$$\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_1}(t_1)}\cdots\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_n}(t_n)}W[J;q(t_b),q(t_a)]|_{J=0}$$
$$= \frac{\hbar}{\mathrm{i}}\langle q_{t_b}|\mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n))|q_{t_a}\rangle_{\mathrm{C}}\,, \tag{1.1.53W}$$

where (1.1.53W) defines the connected part.

Feynman proposed the *mid-point rule* as the approximate formula in the Cartesian coordinate system for $\delta t\cdot L(q_r(t),$ "$q_r(t+\delta t)$") of (1.1.42):

$$\text{``}\delta t\cdot L(q_r(t),\ \text{``}q_r(t+\delta t)\text{''})\text{''}$$
$$= \delta t\cdot L\left(\frac{q_r(t)+q_r(t+\delta t)}{2},\frac{q_r(t+\delta t)-q_r(t)}{\delta t}\right). \tag{1.1.54}$$

In Sect. 1.3, we shall discuss the Feynman path integral formula in the curvilinear coordinate system and the rationale of the mid-point rule. As remarked at the beginning of this chapter, the major results of this section, (1.1.44), (1.1.45), (1.1.50–1.1.52), are not always correct. In Sects. 1.2 and 1.3, we shall gradually make the necessary corrections to the formula derived in Sect. 1.1.

### 1.1.4 The Time-Dependent Schrödinger Equation

The discussions in Sects. 1.1.2 and 1.1.3 are not rigorous at all. In Sect. 1.1.2, our discussion was based upon the quantum analog due to Dirac. In Sect. 1.1.3, our discussion was based upon Feynman's hypothesis (1.1.42). We are not certain whether we are on the right track. In this section, we shall demonstrate the fact that (1.1.42) contains one truth in nonrelativistic quantum mechanics, namely, the time-dependent Schrödinger equation.

We consider a one-dimensional mechanical system described by the following Lagrangian,

$$L\left(q(t),\frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)=\frac{1}{2}m\left(\frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)^2-V(q(t))\,. \tag{1.1.55}$$

Setting $t=t_k$ in (1.1.42), and adopting the mid-point rule, (1.1.54), we obtain the following equation to first order in $\delta t$,

$$\langle q_{t_k+\delta t}|q_{t_k}\rangle$$
$$= A\exp\left\{\frac{\mathrm{i}}{\hbar}\delta t\left[\frac{m}{2}\left(\frac{q_{t_k+\delta t}-q_{t_k}}{\delta t}\right)^2 - V\left(\frac{q_{t_k+\delta t}+q_{t_k}}{2}\right)\right]\right\}. \quad (1.1.56)$$

The wave function $\psi(q_{t_k+\delta t}, t_k+\delta t)$ at time $t = t_k+\delta t$, and position $q = q_{t_k+\delta t}$ is related to the wave function $\psi(q_{t_k}, t_k)$ at time $t = t_k$, and position $q = q_{t_k}$ in the following manner,

$$\begin{aligned}\psi(q_{t_k+\delta t}, t_k+\delta t) &\equiv \langle q_{t_k+\delta t}|\psi\rangle \\ &= \int_{-\infty}^{+\infty}\langle q_{t_k+\delta t}|q_{t_k}\rangle \mathrm{d}q_{t_k}\langle q_{t_k}|\psi\rangle \\ &= \int_{-\infty}^{+\infty}\langle q_{t_k+\delta t}|q_{t_k}\rangle \mathrm{d}q_{t_k}\psi(q_{t_k}, t_k)\,. \end{aligned} \quad (1.1.57)$$

Substituting (1.1.56) into (1.1.57), we have

$$\begin{aligned}&\psi(q_{t_k+\delta t}, t_k+\delta t) \\ &\quad = A\int_{-\infty}^{+\infty}\mathrm{d}q_{t_k}\exp\left[\frac{\mathrm{i}}{\hbar}\delta t\left\{\frac{m}{2}\left(\frac{q_{t_k+\delta t}-q_{t_k}}{\delta t}\right)^2 - V\left(\frac{q_{t_k+\delta t}+q_{t_k}}{2}\right)\right\}\right] \\ &\qquad \times\psi(q_{t_k}, t_k).\end{aligned}$$

Setting

$$q_{t_k+\delta t} = q, \qquad q_{t_k+\delta t} - q_{t_k} = \xi \qquad t_k = t,$$

and hence

$$q_{t_k} = q - \xi,$$

we have

$$\begin{aligned}\psi(q, t+\delta t) = A\exp\left[-\frac{\mathrm{i}}{\hbar}\delta t V(q)\right]\int_{-\infty}^{+\infty}\mathrm{d}\xi\exp\left[\frac{\mathrm{i}m}{\delta t\cdot 2\hbar}\xi^2\right] \\ \times\exp\left[-\frac{\mathrm{i}}{\hbar}\delta t\left\{V\left(q-\frac{\xi}{2}\right) - V(q)\right\}\right]\psi(q-\xi, t)\,. \end{aligned} \quad (1.1.58)$$

In the right-hand side of (1.1.58), we expand the second and third factor in the integrand as follows:

$$\begin{cases} \exp\left[-\dfrac{\mathrm{i}}{\hbar}\delta t\left\{V\left(q-\dfrac{\xi}{2}\right)-V(q)\right\}\right] \\ \qquad = 1-\dfrac{\mathrm{i}\delta t}{2\hbar}\left(\xi V'(q)-\dfrac{\xi^2}{4}V''(q)+\cdots\right)+O((\delta t)^2), \\ \psi(q-\xi,t)=\psi(q,t)-\xi\dfrac{\partial}{\partial q}\psi(q,t)+\dfrac{\xi^2}{2}\dfrac{\partial^2}{\partial q^2}\psi(q,t)-\cdots. \end{cases} \tag{1.1.59}$$

We use the following quasi-Gaussian integral formula

$$\int_{-\infty}^{+\infty} \mathrm{d}\xi \exp\left[\frac{im}{\delta t\cdot 2\hbar}\xi^2\right]\begin{Bmatrix}1\\ \xi\\ \xi^2\\ \xi^3\\ \xi^4\end{Bmatrix}=\begin{Bmatrix}(2\pi\hbar\cdot \mathrm{i}\delta t/m)^{1/2}\\ 0\\ (\hbar\cdot \mathrm{i}\delta t/m)(2\pi\hbar\cdot \mathrm{i}\delta t/m)^{1/2}\\ 0\\ 3(\hbar\cdot \mathrm{i}\delta t/m)^2(2\pi\hbar\cdot \mathrm{i}\delta t/m)^{1/2}\end{Bmatrix}. \tag{1.1.60}$$

Then, we obtain the right-hand side of (1.1.58) as

$$\begin{aligned}\psi(q,t+\delta t) &= \exp\left[-\frac{\mathrm{i}}{\hbar}\delta t V(q)\right] A\left(\frac{2\pi\hbar\cdot \mathrm{i}\delta t}{m}\right)^{1/2} \\ &\quad \times\left\{\psi(q,t)+\frac{\hbar\cdot \mathrm{i}\delta t}{2m}\frac{\partial^2}{\partial q^2}\psi(q,t)+O((\delta t)^2)\right\}.\end{aligned} \tag{1.1.61}$$

From the $0^{\text{th}}$ order matching with respect to $\delta t$, we obtain the normalization constant $A$ from (1.1.61) as

$$A=\left(\frac{2\pi\hbar\cdot \mathrm{i}\delta t}{m}\right)^{-1/2}. \tag{1.1.62}$$

Now, we expand $\psi(q,t+\delta t)$ and $\exp[-(\mathrm{i}/\hbar)\delta t V(q)]$ in (1.1.61) to first order in $\delta t$. After a little algebra, we obtain from (1.1.61),

$$\begin{aligned}&\delta t\cdot\frac{\partial}{\partial t}\psi(q,t)+O((\delta t)^2) \\ &\quad = -\frac{\mathrm{i}}{\hbar}\delta t\cdot\left\{V(q)-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\right\}\psi(q,t)+O((\delta t)^2).\end{aligned} \tag{1.1.63}$$

Multiplying $\mathrm{i}\hbar/\delta t$ on both sides of (1.1.63) and taking the limit $\delta t\to 0$, we obtain the time-dependent Schrödinger equation,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(q,t)=\left[\frac{1}{2m}\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q}\right)^2+V(q)\right]\psi(q,t). \tag{1.1.64}$$

In this way, we have demonstrated the fact that Feynman's hypothesis (1.1.42) contains one piece of truth, i.e., the time-dependent Schrödinger equation.

In so far as the derivation of the time-dependent Schrödinger equation (1.1.64) is concerned, we can use the approximation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}q(t_k)\right)^2 \simeq \left(\frac{q_{t_k+\delta t}-q_{t_k}}{\delta t}\right)^2 , \tag{1.1.65}$$

to obtain the correct result. However, when we calculate the expectation value of $\mathrm{d}\hat{q}(t)/\mathrm{d}t$ raised to some power, for example, the kinetic energy

$$\frac{1}{2}m\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{q}(t)\right)^2 ,$$

we get a divergent result as $\delta t \to 0$, if we use the approximation (1.1.65). Feynman proposed the approximate formula for $(\mathrm{d}q(t)/\mathrm{d}t)^2$ in (1.1.42) of the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}q(t_k)\right)^2 \simeq \left(\frac{q_{t_k+\delta t}-q_{t_k}}{\delta t}\right)\left(\frac{q_{t_k}-q_{t_k-\delta t}}{\delta t}\right) . \tag{1.1.66}$$

Lastly, we state the physical interpretation of (1.1.56) and (1.1.57). We regard these equations as statements of Huygen's principle for the wave function $\psi(q,t)$. When the wave function $\psi(q_{t_k},t_k)$ on the "surface" $S_{t_k}$, which is constituted by all possible values of the position $q_{t_k}$ at time $t_k$, is given, the values of the wave function $\psi(q_{t_k+\delta t},t_k+\delta t)$ at the neighboring position $q_{t_k+\delta t}$ at time $t_k+\delta t$ are given by the sum of the contributions from the values of $\psi(q_{t_k},t_k)$ on $S_{t_k}$. Each contribution is retarded with the phase factor $I(q_{t_k+\delta t},q_{t_k})/\hbar$, proportional to the action functional $I(q_{t_k+\delta t},q_{t_k})$ which is the time integral of the Lagrangian along the classical path

$$\left\{q^{\mathrm{cl}}(t), t_k+\delta t \geq t \geq t_k\right\} .$$

Huygen's principle in wave optics requires the specification of

$$\psi(q_{t_k},t_k) \quad \text{and} \quad \frac{\partial}{\partial t_k}\psi(q_{t_k},t_k) \quad \text{on} \;\; S_{t_k},$$

due to the fact that the wave equation in optics is second order in the partial derivative with respect to time $t$, whereas in quantum mechanics the specification of

$$\psi(q_{t_k},t_k) \quad \text{on} \;\; S_{t_k}$$

alone is required due to the fact that the time-dependent Schrödinger equation, (1.1.64), is first order in the partial derivative with respect to time $t$.

### 1.1.5 The Principle of Superposition and the Composition Law

In this section, we reconsider the method of path integral quantization from the principle of superposition and the composition law. As a result, we shall find that the expression (1.1.46),

$$\int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] = \int_{-\infty}^{+\infty} \mathrm{d}q' \int_{q_{\mathrm{II}}(t)=q'}^{q_{\mathrm{II}}(t_b)=q_{t_b}} \mathcal{D}[q_{\mathrm{II}}(t)] \int_{q_{\mathrm{I}}(t_a)=q_{t_a}}^{q_{\mathrm{I}}(t)=q'} \mathcal{D}[q_{\mathrm{I}}(t)],$$

in path integral quantization is nothing but a statement of the principle of superposition and the composition law in quantum mechanics. Furthermore, we shall find the notion of the probability amplitude,

$$\varphi\left[\{q_r(t); t_b \geq t \geq t_a\}_{r=1}^{f}\right],$$

associated with the path $\{q_r(t); t_b \geq t \geq t_a\}_{r=1}^{f}$ connecting $q_r(t_a)$ and $q_r(t_b)$; $\varphi$[path] is given by

$$\begin{aligned}\varphi\left[\{q_r(t); t_b \geq t \geq t_a\}_{r=1}^{f}\right] &\propto \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t L\left(q_r(t), \frac{\mathrm{d}}{\mathrm{d}t}q_r(t)\right)\right] \\ &= \exp\left[\frac{\mathrm{i}}{\hbar} I[q; q_{t_b}, q_{t_a}]\right].\end{aligned} \tag{1.1.67}$$

In the right-hand side of (1.1.67), the time integral is taken along the path specified in the left-hand side. We note that (1.1.67) is an extension of the probability amplitude, (1.1.42), for the infinitesimal time interval, $[t, t + \delta t]$, to the case of the finite time interval, $[t_a, t_b]$.

We consider a *gedanken* experiment of an electron beam, ejected by an electron gun $G$, going through the double holes $H_1$ and $H_2$ on the screen $S$, and detected by a detector $D$ on the detector screen. We let $\varphi_{H_1}$ be the probability amplitude of the electron, $G \rightarrow H_1 \rightarrow D$, and $\varphi_{H_2}$ be the probability amplitude of the electron, $G \rightarrow H_2 \rightarrow D$. We obtain the total probability amplitude, $\varphi_{D\leftarrow G}$, of the electron, $G \rightarrow D$, from the principle of superposition as

$$\varphi_{D\leftarrow G} = \varphi_{H_1} + \varphi_{H_2}. \tag{1.1.68}$$

We know the probability amplitudes, $\varphi_{H_1}$ and $\varphi_{H_2}$, are given by the composition law of the probability amplitudes as

$$\varphi_{H_i} = \varphi_{D\leftarrow H_i} \cdot \varphi_{H_i\leftarrow G}, \qquad i = 1, 2. \tag{1.1.69}$$

Next, we make infinitely many holes $\{H_i\}_{i=1}^{\infty}$ on the screen $S$, and again use the principle of superposition and the composition law, (1.1.68) and (1.1.69), to obtain the total probability amplitude, $\varphi_{D\leftarrow G}$,

$$\varphi_{D\leftarrow G} = \sum_{\{H_i\}_{i=1}^{\infty}} \varphi_{H_i} = \sum_{\{H_i\}_{i=1}^{\infty}} \varphi_{D\leftarrow H_i} \cdot \varphi_{H_i\leftarrow G} \,. \tag{1.1.70}$$

Lastly, we erect infinitely many screens $S_{\mathrm{I}}$ and $S_{\mathrm{II}}$, between $G$ and $S$, and $S$ and $D$. With repeated use of (1.1.68–1.1.70), we obtain the total probability amplitude, $\varphi_{D\leftarrow G}$, as

$$\varphi_{D\leftarrow G} = \sum_{\substack{\text{all paths}\\ D\leftarrow G}} \varphi[\text{particular path: } D \leftarrow G] \,. \tag{1.1.71}$$

Comparing the result, (1.1.71), with the path integral representation of the transformation function $\langle q_{t_b} | q_{t_a} \rangle$, (1.1.45), and the description of the path integral symbol, (1.1.46), we reach the notion of probability amplitude,

$$\varphi\left[\{q_r(t); t_b \geq t \geq t_a\}_{r=1}^{f}\right] , \tag{1.1.67}$$

associated with the path,

$$\{q_r(t); t_b \geq t \geq t_a\}_{r=1}^{f} .$$

In this way, we realize that the path integral representation of the transformation function $\langle q_{t_b} | q_{t_a} \rangle$, (1.1.45), embodies the principle of superposition, (1.1.68), and the composition law, (1.1.69), under the identification, (1.1.67), and is written in an elegant notation with the path integral symbol

$$\int \mathcal{D}[q(t)]$$

in the sense of (1.1.46). When we apply the conclusion for $\varphi_{D\leftarrow G}$, (1.1.71), to each of $\varphi_{H_i\leftarrow G}$ and $\varphi_{D\leftarrow H_i}$ of (1.1.70), we find

$$\begin{aligned} &\sum_{\substack{\text{all paths}\\ D\leftarrow G}} \varphi[\text{particular path: } D \leftarrow G] \\ &\qquad = \sum_{\{H_i\}_{i=1}^{\infty}} \sum_{\substack{\text{all paths}\\ D\leftarrow H_i}} \varphi[\text{particular path: } D \leftarrow H_i] \\ &\qquad\quad \times \sum_{\substack{\text{all paths}\\ H_i\leftarrow G}} \varphi[\text{particular path: } H_i \leftarrow G] \,. \end{aligned} \tag{1.1.72}$$

In view of the probability amplitude $\varphi[\text{path}]$, (1.1.67), the additivity of the action functional $I[q; q(t_b), q(t_a)]$, and the meaning of the path integral symbol, (1.1.46), we realize that (1.1.72), which represents the principle of superposition and the composition law of the probability amplitudes, is equivalent to (1.1.47).

From all the considerations we have discussed so far, we understand that

(1) Feynman path integral quantization is $c$-number quantization based on the Lagrangian formalism of analytical mechanics,

and,

(2) Feynman path integral quantization contains the principle of superposition and the composition law of the transition probability amplitude *ab initio*.

## 1.2 Path Integral Representation of Quantum Mechanics in the Hamiltonian Formalism

In this section we discuss the standard method of deriving the path integral formula from the Hamiltonian formalism of quantum mechanics. As a result, we shall discover that the conclusions of the previous section, (1.1.45), (1.1.50–1.1.52), are not always correct.

In Sect. 1.2.1, we briefly review quantum mechanics in the Hamiltonian formalism and establish the notation. In Sects. 1.2.2 and 1.2.3, we derive the phase space path integral formula for the transformation function,

$$\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle,$$

and the matrix element of the time-ordered product,

$$\langle q_{t_b}, t_b | \mathrm{T}(\hat{q}(t_1) \cdots \hat{q}(t_n)) | q_{t_a}, t_a \rangle,$$

respectively. In Sect. 1.2.4, we derive the vacuum state wave function in order to take the limit $t_a \to -\infty$ and $t_b \to +\infty$. In Sect. 1.2.5, we derive the phase space path integral formula for the generating functional, $Z[J]$, of the Green's functions. In Sect. 1.2.6, we perform the functional Gaussian integration with respect to the canonical momentum $p(t)$, when the Hamiltonian $H(q(t), p(t))$ is given by the quadratic form in the canonical momentum $p(t)$. When the kernel of the quadratic form is a constant matrix, we show that the formula derived in Sects. 1.2.2–1.2.5 is given by the configuration space path integral formula in the Lagrangian formalism of the previous section. When the kernel of the quadratic form is a $q(t)$-dependent matrix, we find that the functional Gaussian integration stated above generates a $q(t)$-dependent determinant factor in the integrand of the configuration space path integral formula, and we have to make a requisite correction to the conclusions of the previous section: we have to replace the original Lagrangian with the effective Lagrangian which takes the presence of the $q(t)$-dependent determinant factor into consideration. In this manner, we shall see the power and the limitation of the discussion of the previous section based on the Lagrangian formalism.

As for the operator-ordering problem, we still evade the discussion with the notion of a well-ordered operator and, in the next section, we discuss the problem with the notion of the Weyl correspondence.

### 1.2.1 Review of Quantum Mechanics in the Hamiltonian Formalism

We consider a canonical quantum mechanical system described by the pair of canonical variables $\{\hat{q}_r(t), \hat{p}_r(t)\}_{r=1}^{f}$ and the Hamiltonian $H(\hat{q}_r(t), \hat{p}_r(t))$ in the Heisenberg picture,

$$\hat{q}_r(t), \quad \hat{p}_s(t), \quad r, s = 1, \dots, f; \qquad H = H(\hat{q}_r(t), \hat{p}_r(t)) \,. \tag{1.2.1}$$

We have the equal-time canonical commutation relations,

$$\begin{cases} [\hat{q}_r(t), \hat{q}_s(t)] = [\hat{p}_r(t), \hat{p}_s(t)] = 0, \\ [\hat{q}_r(t), \hat{p}_s(t)] = \mathrm{i}\hbar\delta_{r,s}. \end{cases} \qquad r, s = 1, \dots, f. \tag{1.2.2}$$

The Heisenberg equation of motion for an arbitrary operator $\hat{O}(t)$ in the Heisenberg picture is given by

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t}\hat{O}(t) = [\hat{O}(t), H(\hat{q}_r(t), \hat{p}_r(t))] \,. \tag{1.2.3}$$

When we take $\hat{q}_r(t)$ and $\hat{p}_r(t)$ as $\hat{O}(t)$, we can immediately integrate (1.2.3), and obtain

$$\begin{cases} \hat{q}_r(t) = \exp\left[\frac{\mathrm{i}}{\hbar}H(\hat{q}_s(t), \hat{p}_s(t))t\right] \hat{q}_r(0) \exp\left[-\frac{\mathrm{i}}{\hbar}H(\hat{q}_s(t), \hat{p}_s(t))t\right], \\ \hat{p}_r(t) = \exp\left[\frac{\mathrm{i}}{\hbar}H(\hat{q}_s(t), \hat{p}_s(t))t\right] \hat{p}_r(0) \exp\left[-\frac{\mathrm{i}}{\hbar}H(\hat{q}_s(t), \hat{p}_s(t))t\right]. \end{cases} \tag{1.2.4}$$

We let $|q, t\rangle$ ($|p, t\rangle$) be the eigenket of the operator $\hat{q}(t)$ ($\hat{p}(t)$) belonging to the eigenvalue $q_r$ ($p_r$),

$$\begin{cases} \hat{q}_r(t)|q, t\rangle = q_r|q, t\rangle, \\ \hat{p}_r(t)|p, t\rangle = p_r|p, t\rangle. \end{cases} \qquad r = 1, \dots, f, \tag{1.2.5}$$

From (1.2.4) and (1.2.5), we obtain the time dependence of $|q, t\rangle$,

$$\begin{cases} |q, t\rangle = \exp\left[\frac{\mathrm{i}}{\hbar}H(\hat{q}_r(t), \hat{p}_r(t))t\right] |q, 0\rangle, \\ |q, t\rangle = \exp\left[\frac{\mathrm{i}}{\hbar}H(\hat{q}_r(t), \hat{p}_r(t))(t - t')\right] |q, t'\rangle. \end{cases} \tag{1.2.6}$$

From (1.2.6), we learn that the eigenkets $|q, t\rangle$ and $|p, t\rangle$ are a "rotating coordinate" system. At first sight, however, the time dependence of $|q, t\rangle$ appears to be wrong. In order to show that the time dependence of (1.2.6) is correct, we consider the wave function $\psi(q,t)$.

In the Schrödinger picture, we have the time-dependent state vector $|\psi(t)\rangle_{\mathrm{S}}$ which satisfies the Schrödinger equation,

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle_{\mathrm{S}} = H(\hat{q}_r(t),\hat{p}_r(t))|\psi(t)\rangle_{\mathrm{S}}.$$

The formal solution of this equation is given by

$$|\psi(t)\rangle_{\mathrm{S}} = \exp\left[-\frac{\mathrm{i}}{\hbar}H(\hat{q}_r(t),\hat{p}_r(t))t\right]|\psi(0)\rangle_{\mathrm{S}}.$$

Choosing $t=0$ as the time when the Schrödinger picture coincides with the Heisenberg picture, we have the wave function $\psi_{\mathrm{S}}(q,t)$ in the Schrödinger picture as

$$\psi_{\mathrm{S}}(q,t) \equiv \langle q,0|\psi(t)\rangle_{\mathrm{S}} = \langle q,0|\exp\left[-\frac{\mathrm{i}}{\hbar}H(\hat{q}_r(t),\hat{p}_r(t))t\right]|\psi(0)\rangle\,. \tag{1.2.7}$$

On the other hand, in the Heisenberg picture, we have the time-independent state vector,

$$|\psi\rangle_{\mathrm{H}} \equiv |\psi(0)\rangle.$$

But we can define the wave function $\psi_{\mathrm{H}}(q,t)$ by the projection of $|\psi\rangle_{\mathrm{H}}$ onto the "rotating coordinate" system $\langle q,t|$, i.e.,

$$\psi_{\mathrm{H}}(q,t) \equiv \langle q,t|\psi\rangle_{\mathrm{H}} = \langle q,0|\exp\left[-\frac{\mathrm{i}}{\hbar}H(\hat{q}_r(t),\hat{p}_r(t))t\right]|\psi(0)\rangle\,. \tag{1.2.8}$$

The two wave functions given by (1.2.7) and (1.2.8) coincide with each other as they should:

$$\psi(q,t) = \psi_{\mathrm{S}}(q,t) = \psi_{\mathrm{H}}(q,t)\,. \tag{1.2.9}$$

The time dependence of the Schrödinger wave function originates from the time dependence of the state vector $|\psi(t)\rangle_{\mathrm{S}}$ which satisfies the Schrödinger equation, whereas the time dependence of the Heisenberg wave function originates from the time-dependent "rotating coordinate" system $\langle q,t|$. The Schrödinger equation can be regarded as the constraint on $\langle q,t|\,\psi\rangle_{\mathrm{H}}$. With these considerations, we are assured that the time dependence of $|q,t\rangle$ as expressed by (1.2.6) is correct.

We record here several formulas which become necessary in Sect. 1.2.2:

$$\langle q,t|p,t\rangle \ = \langle q|p\rangle = (2\pi\hbar)^{-f/2}\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{r=1}^{f}p_rq_r\right]. \tag{1.2.10}$$

This follows from the equations

$$\langle q|\hat{p}_r|p\rangle = p_r\langle q|p\rangle \;=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q_r}\langle q|p\rangle\,, \tag{1.2.11}$$

$$\langle q'|q\rangle = \delta^f(q'-q).$$

With the assumption that the Hamiltonian $H(\hat{q}_r(t),\hat{p}_r(t))$ is a "well-ordered" operator, i.e., all $\hat{q}'$s to the left of all $\hat{p}'$s, we have the identity

$$\begin{aligned}\langle q',t'|p,t\rangle &= \left\langle q',t\left|\exp\left[-\frac{\mathrm{i}}{\hbar}H(\hat{q}_r(t),\hat{p}_r(t))(t'-t)\right]\right|p,t\right\rangle \\ &= \exp\left[-\frac{\mathrm{i}}{\hbar}H(q'_r,p_r)(t'-t)\right]\cdot\langle q',t|p,t\rangle \\ &= (2\pi\hbar)^{-f/2}\exp\left[-\frac{\mathrm{i}}{\hbar}H(q'_r,p_r)(t'-t)+\frac{\mathrm{i}}{\hbar}\sum_{r=1}^{f}p_r q'_r\right]. \end{aligned}\tag{1.2.12}$$

As the resolution of identity in phase space, we have,

$$\int_{-\infty}^{+\infty}\prod_{r=1}^{f}\mathrm{d}q_r|q,t\rangle\langle q,t| = \int_{-\infty}^{+\infty}\prod_{r=1}^{f}\mathrm{d}p_r|p,t\rangle\langle p,t| = 1\,. \tag{1.2.13}$$

As for the transformation function $\langle q',t'|q,t\rangle$, with the use of (1.2.10), (1.2.12) and (1.2.13), we have

$$\begin{aligned}\langle q',t'|q,t\rangle &= \int\langle q',t'|p,t\rangle\prod_{r=1}^{f}\mathrm{d}p_r\langle p,t|q,t\rangle \\ &= (2\pi\hbar)^{-f}\int\prod_{r=1}^{f}\mathrm{d}p_r \\ &\qquad\times\exp\left[\frac{\mathrm{i}}{\hbar}\left\{\sum_{r=1}^{f}p_r(q'_r-q_r)-H(q'_r,p_r)(t'-t)\right\}\right], \end{aligned}\tag{1.2.14}$$

which is correct when the Hamiltonian $H(\hat{q}_r(t),\hat{p}_r(t))$ is a "well-ordered" operator.

### 1.2.2 Phase Space Path Integral Representation of the Transformation Function

In order to obtain the phase space path integral representation of the transformation function

$$\langle q_{t_b},t_b|q_{t_a},t_a\rangle,$$

we divide the time interval $[t_a,t_b]$ into $n$ equal subintervals $[t_k,t_{k+1}]$, where

$$t_b \equiv t_n > t_{n-1} > \cdots > t_1 > t_0 \equiv t_a,$$

$$\delta t = \frac{t_b - t_a}{n}, \quad t_k = t_a + k\delta t, \quad k = 0, 1, \ldots, n-1, n,$$

and let the values of $q_r(t)$ at $t = t_k$ be

$$q_{r,t_k} \equiv q_r(t_k).$$

We apply the resolution of identity, (1.2.13), at each time $t = t_k$. From (1.2.10), (1.2.12) and (1.2.14), we obtain

$$\begin{aligned}
&\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle \\
&= \int_{q_{t_0}=q_{t_a}}^{q_{t_n}=q_{t_b}} \prod_{k=1}^{n-1} \mathrm{d}^f q_{t_k} \prod_{k=0}^{n-1} \langle q_{t_{k+1}}, t_{k+1} | q_{t_k}, t_k \rangle \\
&= \int_{q_{t_0}=q_{t_a}}^{q_{t_n}=q_{t_b}} \prod_{k=1}^{n-1} \mathrm{d}^f q_{t_k} \int \prod_{k=0}^{n-1} \mathrm{d}^f p_{t_k} \\
&\quad \times \prod_{k=0}^{n-1} \langle q_{t_{k+1}}, t_{k+1} | p_{t_k}, t_k \rangle \langle p_{t_k}, t_k | q_{t_k}, t_k \rangle \\
&= \int_{q_{t_0}=q_{t_a}}^{q_{t_n}=q_{t_b}} \prod_{k=1}^{n-1} \frac{\mathrm{d}^f q_{t_k}}{\sqrt{(2\pi\hbar)^f}} \int \prod_{k=0}^{n-1} \frac{\mathrm{d}^f p_{t_k}}{\sqrt{(2\pi\hbar)^f}} \\
&\quad \times \exp\left[\sum_{k=0}^{n-1} \left\{ \sum_{r=1}^{f} p_{r,t_k+\delta t}(q_{r,t_k+\delta t} - q_{r,t_k}) - H(q_{t_{k+1}}, p_{t_k})\delta t \right\}\right], \qquad (1.2.15)
\end{aligned}$$

where we have assumed that the Hamiltonian $H(\hat{q}_r(t), \hat{p}_r(t))$ is "well-ordered". We further assume that the operators $\hat{q}(t)$ and $\hat{p}(t)$, have continuous eigenvalues, $q(t)$ and $p(t)$ parametrized by $t$. Then, in the limit $n \to \infty$, we have the following expression for the exponent in the integrand of (1.2.15):

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=0}^{n-1} \left\{ \sum_{r=1}^{f} p_{r,t_k+\delta t}(q_{r,t_k+\delta t} - q_{r,t_k}) - H(q_{t_{k+1}}, p_{t_k})\delta t \right\} \\
&\quad = \lim_{n\to\infty} \sum_{k=0}^{n-1} \delta t \left\{ \sum_{r=1}^{f} p_{r,t_k+\delta t} \left( \frac{q_{r,t_k+\delta t} - q_{r,t_k}}{\delta t} \right) - H(q_{t_{k+1}}, p_{t_k}) \right\} \\
&\quad = \int_{t_a}^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^{f} p_r(t) \frac{\mathrm{d}}{\mathrm{d}t} q_r(t) - H(q(t), p(t)) \right\}. \qquad (1.2.16)
\end{aligned}$$

Thus, in the limit $n \to \infty$ in (1.2.15), we obtain the phase space path integral representation of the transformation function:

$$\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle$$

$$= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)]$$

$$\times \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{t_a}^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^{f} p_r(t) \frac{\mathrm{d}}{\mathrm{d}t} q_r(t) - H(q(t), p(t)) \right\} \right]. \qquad (1.2.17)$$

Here, we note the definition of the phase space path integral symbol,

$$\int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] \equiv \lim_{n\to\infty} \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \prod_{k=1}^{n-1} \frac{\mathrm{d}^f q_{t_k}}{\sqrt{(2\pi\hbar)^f}}$$

$$\times \int \prod_{k=0}^{n-1} \frac{\mathrm{d}^f p_{t_k}}{\sqrt{(2\pi\hbar)^f}}. \qquad (1.2.18)$$

We remark that in the phase space path integral representation of the transformation function, (1.2.17), the momentum integration is unconstrained. We also remark that in (1.2.16) and (1.2.17), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} q_r(t) = \frac{q_r(t+\delta t) - q_r(t)}{\delta t}, \qquad (1.2.19)$$

to first order in $\delta t$. We note that $\mathrm{d}q_r(t)/\mathrm{d}t$ becomes a function of $q_s(t)$ and $p_s(t)$, $(s = 1, \ldots, f)$, as a result of the momentum integration in (1.2.17). We shall discuss this point in some detail in Sect. 1.2.6.

### 1.2.3 Matrix Element of a Time-Ordered Product

In this subsection, we derive the phase space path integral representation of the matrix element of the time-ordered product of $\hat{q}(t)$,

$$\langle q_{t_b}, t_b | \mathrm{T}(\hat{q}_{r_1}(t_1) \cdots \hat{q}_{r_n}(t_n)) | q_{t_a}, t_a \rangle$$

$$= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] q_{r_1}(t_1) \cdots q_{r_n}(t_n)$$

$$\times \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{t_a}^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^{f} p_r(t) \frac{\mathrm{d}}{\mathrm{d}t} q_r(t) - H(q(t), p(t)) \right\} \right]. \qquad (1.2.20)$$

The principle of superposition and the composition law of the probability amplitude in the phase space path integral representation is given by

$$\int_{q(t_a)=q_{t_a}}^{q(t_{t_b})=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)]$$

$$= \int_{-\infty}^{+\infty} \mathrm{d}^f q_t \int_{q_{\mathrm{II}}(t)=q_t}^{q_{\mathrm{II}}(t_b)=q_{t_b}} \mathcal{D}[q_{\mathrm{II}}(t)] \int \mathcal{D}[p_{\mathrm{II}}(t)]$$

$$\times \int_{q_{\mathrm{I}}(t_a)=q_{t_a}}^{q_{\mathrm{I}}(t)=q_t} \mathcal{D}[q_{\mathrm{I}}(t)] \int \mathcal{D}[p_{\mathrm{I}}(t)] \,. \tag{1.2.21}$$

In the first place, we try to obtain the matrix element of $\hat{q}_r(t)$,

$$\langle q_{t_b}, t_b | \hat{q}_r(t) | q_{t_a}, t_a \rangle,$$

in the phase space path integral representation. We apply the resolution of identity, (1.2.13), in the $q$-representation at time $t$, obtaining

$$\langle q_{t_b}, t_b | \hat{q}_r(t) | q_{t_a}, t_a \rangle = \int_{-\infty}^{+\infty} \mathrm{d}^f q_t \langle q_{t_b}, t_b | q_t, t \rangle q_{r,t} \langle q_t, t | q_{t_a}, t_a \rangle \,. \tag{1.2.22}$$

For each transformation function on the right-hand side of (1.2.22), we use the phase space path integral formula, (1.2.17), obtaining

$$\begin{aligned}
&\langle q_{t_b}, t_b | \hat{q}_r(t) | q_{t_a}, t_a \rangle \\
&\quad = \int_{-\infty}^{+\infty} \mathrm{d}^f q_t \int_{q_{\mathrm{II}}(t)=q_t}^{q_{\mathrm{II}}(t_b)=q_{t_b}} \mathcal{D}[q_{\mathrm{II}}(t)] \int \mathcal{D}[p_{\mathrm{II}}(t)] \\
&\qquad \times \exp\left[\frac{\mathrm{i}}{\hbar} \int_t^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^f p_{\mathrm{II},r}(t) \frac{\mathrm{d}}{\mathrm{d}t} q_{\mathrm{II},r}(t) - H(q_{\mathrm{II}}(t), p_{\mathrm{II}}(t)) \right\}\right] \\
&\qquad \times q_{r,t} \int_{q_{\mathrm{I}}(t_a)=q_{t_a}}^{q_{\mathrm{I}}(t)=q_t} \mathcal{D}[q_{\mathrm{I}}(t)] \int \mathcal{D}[p_{\mathrm{I}}(t)] \\
&\qquad \times \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t_a}^{t} \mathrm{d}t \left\{ \sum_{r=1}^f p_{\mathrm{I},r}(t) \frac{\mathrm{d}}{\mathrm{d}t} q_{\mathrm{I},r}(t) - H(q_{\mathrm{I}}(t), p_{\mathrm{I}}(t)) \right\}\right].
\end{aligned} \tag{1.2.23}$$

We now apply the principle of superposition and the composition law, (1.2.21), to the right-hand side of (1.2.23), obtaining

$$\begin{aligned}
&\langle q_{t_b}, t_b | \hat{q}_r(t) | q_{t_a}, t_a \rangle \\
&\quad = \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] q_r(t) \\
&\qquad \times \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t_a}^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^f p_r(t) \frac{\mathrm{d}}{\mathrm{d}t} q_r(t) - H(q(t), p(t)) \right\}\right].
\end{aligned} \tag{1.2.24}$$

This is the phase space path integral representation of $\langle q_{t_b}, t_b | \hat{q}_r(t) | q_{t_a}, t_a \rangle$.

Next, we consider the phase space path integral representation of the matrix element

$$\langle q_{t_b}, t_b | \mathrm{T}(\hat{q}_{r_1}(t_1)\hat{q}_{r_2}(t_2)) | q_{t_a}, t_a \rangle .$$

We first consider the case $t_2 > t_1$. We apply the resolution of identity, (1.2.13), in the $q$-representation at time $t_2$, obtaining

$$\begin{aligned} &\langle q_{t_b}, t_b | \hat{q}_{r_2}(t_2)\hat{q}_{r_1}(t_1) | q_{t_a}, t_a \rangle \\ &= \int_{-\infty}^{+\infty} \mathrm{d}^f q_{t_2} \langle q_{t_b}, t_b | q_{t_2}, t_2 \rangle q_{r_2}(t_2) \langle q_{t_2}, t_2 | \hat{q}_{r_1}(t_1) | q_{t_a}, t_a \rangle . \end{aligned} \tag{1.2.25}$$

We apply the phase space path integral formula, (1.2.17) and (1.2.24), and use the principle of superposition and the composition law, (1.2.21), to the right-hand side of (1.2.25), obtaining

$$\begin{aligned} &\langle q_{t_b}, t_b | \hat{q}_{r_2}(t_2)\hat{q}_{r_1}(t_1) | q_{t_a}, t_a \rangle \\ &= \int_{-\infty}^{+\infty} \mathrm{d}^f q_{t_2} \int_{q_{\mathrm{II}}(t_2)=q_{t_2}}^{q_{\mathrm{II}}(t_b)=q_{t_b}} \mathcal{D}[q_{\mathrm{II}}(t)] \int \mathcal{D}[p_{\mathrm{II}}(t)] \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_2}^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^{f} p_{\mathrm{II},r}(t)\frac{\mathrm{d}}{\mathrm{d}t} q_{\mathrm{II},r}(t) - H(q_{\mathrm{II}}(t), p_{\mathrm{II}}(t)) \right\}\right] q_{r_2}(t_2) \\ &\quad \times \int_{q_{\mathrm{I}}(t_a)=q_{t_a}}^{q_{\mathrm{I}}(t_2)=q_{t_2}} \mathcal{D}[q_{\mathrm{I}}(t)] \int \mathcal{D}[p_{\mathrm{I}}(t)] q_{r_1}(t_1) \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_2} \mathrm{d}t \left\{ \sum_{r=1}^{f} p_{\mathrm{I}}(t)\frac{\mathrm{d}}{\mathrm{d}t} q_{\mathrm{I}}(t) - H(q_{\mathrm{I}}(t), p_{\mathrm{I}}(t)) \right\}\right] \\ &= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] q_{r_2}(t_2) q_{r_1}(t_1) \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^{f} p_r(t)\frac{\mathrm{d}}{\mathrm{d}t} q_r(t) - H(q(t), p(t)) \right\}\right] . \end{aligned} \tag{1.2.26}$$

We next consider the case $t_1 > t_2$, obtaining the same result as (1.2.26). Hence, we have the formula,

$$\begin{aligned} &\langle q_{t_b}, t_b | \mathrm{T}(\hat{q}_{r_1}(t_1)\hat{q}_{r_2}(t_2)) | q_{t_a}, t_a \rangle \\ &= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] q_{r_1}(t_1) q_{r_2}(t_2) \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^{f} p_r(t)\frac{\mathrm{d}}{\mathrm{d}t} q_r(t) - H(q(t), p(t)) \right\}\right] . \end{aligned} \tag{1.2.27}$$

For general $n \geq 3$, we can complete the proof of (1.2.20) by mathematical induction with repeated use of the principle of superposition and the composition law of the probability amplitude, (1.2.21).

A noteworthy point of formulas (1.2.20) and (1.2.27) is the fact that we do not have to time-order the $c$-number function $q_{r_1}(t_1)\dots\ q_{r_n}(t_n)$ explicitly by hand, in the right-hand sides. The time-ordering is done automatically by the definition of the phase space path integration. Also, the right-hand sides of (1.2.20) and (1.2.27) are well defined for a coincident time argument ($t_i = t_j$, $i \neq j$). Furthermore, (1.2.20) and (1.2.27) are a linear functional of $q_{r_i}(t_i)$, and thus the time derivative $\mathrm{d}/\mathrm{d}t_i$ and the phase space path integral $\int \mathcal{D}[q(t)] \int \mathcal{D}[p(t)]$ commute. Hence, the quantities defined by the right-hand sides of (1.2.20) and (1.2.27) are not the matrix elements of the T-product, but those of the canonical T*-product which is defined for quantum mechanics by the following equations,

$$\mathrm{T}^*\left(\frac{\mathrm{d}}{\mathrm{d}t_1}\hat{O}_1(t_1)\cdots\hat{O}_n(t_n)\right) \equiv \frac{\mathrm{d}}{\mathrm{d}t_1}\mathrm{T}^*(\hat{O}_1(t_1)\cdots\hat{O}_n(t_n)),$$
$$\mathrm{T}^*(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n)) \equiv \mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n)),$$

with

$$\mathrm{T}(\hat{q}_{r_1}(t_1)\hat{q}_{r_2}(t_2)) \equiv \theta(t_1-t_2)\hat{q}_{r_1}(t_1)\hat{q}_{r_2}(t_2) + \theta(t_2-t_1)\hat{q}_{r_2}(t_2)\hat{q}_{r_1}(t_1).$$

We define the generating functionals $Z[J;q(t_b),q(t_a)]$ and $W[J;q(t_b),q(t_a)]$ of the matrix element of the time-ordered product of the operator $\hat{q}(t)$ and its connected part (to be defined shortly) by

$$\begin{aligned}
&Z[J;q(t_b),q(t_a)] \\
&\quad \equiv \exp\left[\frac{\mathrm{i}}{\hbar}W[J;q(t_b),q(t_a)]\right] \\
&\quad \equiv \left\langle q_{t_b}, t_b \middle| \mathrm{T}\left(\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}\mathrm{d}t\sum_{r=1}^{f}\hat{q}_r(t)J_r(t)\right]\right)\middle| q_{t_a}, t_a\right\rangle \\
&\quad \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{i}}{\hbar}\right)^n\int_{t_a}^{t_b}\mathrm{d}t_1\cdots\mathrm{d}t_n\sum_{r_1,\cdots,r_n=1}^{f}J_{r_1}(t_1)\cdots J_{r_n}(t_n) \\
&\qquad \times\langle q_{t_b},t_b|\mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n))|q_{t_a},t_a\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] \\
&\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t \left\{\sum_{r=1}^{f} p_r(t)\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - H(q(t),p(t)) \right.\right. \\
&\qquad\qquad\qquad \left.\left. + \sum_{r=1}^{f} q_r(t)J_r(t)\right\}\right] .
\end{aligned}
\tag{1.2.28}
$$

The last line of (1.2.28) follows from (1.2.20). We have the formulas

$$
\begin{aligned}
&\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_1}(t_1)} \cdots \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_n}(t_n)} Z[J; q(t_b), q(t_a)]|_{J=0} \\
&\qquad = \langle q_{t_b}, t_b | \mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n)) | q_{t_a}, t_a\rangle ,
\end{aligned}
\tag{1.2.29Z}
$$

and

$$
\begin{aligned}
&\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_1}(t_1)} \cdots \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_n}(t_n)} W[J; q(t_b), q(t_a)]|_{J=0} \\
&\qquad = \frac{\hbar}{\mathrm{i}}\langle q_{t_b}, t_b | \mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n)) | q_{t_a}, t_a\rangle_{\mathrm{C}} ,
\end{aligned}
\tag{1.2.29W}
$$

where (1.2.29W) defines the connected part of the matrix element of the T-product.

### 1.2.4 Wave Function of the Vacuum

In the next subsection, we shall discuss the phase space path integral representation of the generating functional, $Z[J]$ and $W[J]$, of the Green's functions, which are defined as the vacuum expectation value of the time-ordered product of $\hat{q}_r(t)$, and its connected part. For this purpose, it is not sufficient to take the limit $t_a \to -\infty$ and $t_b \to +\infty$ in (1.2.29Z) and (1.2.29W), since the integral

$$
\int \mathcal{D}[q(t)]
$$

is still constrained by $q(\pm\infty) = q_{\pm\infty}$. As a preparation to remove these constraints, we derive the wave functions,

$$
\langle q_{\pm\infty}, \pm\infty | 0, {}^{\mathrm{out}}_{\mathrm{in}} \rangle ,
$$

of the vacuum states $|0, {}^{\mathrm{out}}_{\mathrm{in}}\rangle$ in this subsection.

We adopt the adiabatic switching hypothesis of the interaction. We have the following equations as $t \to \pm\infty$,

$$\hat{q}_r(t) = \hat{a}_r(t) + \hat{a}_r^\dagger(t) \to \hat{a}_r^{\substack{\text{out}\\\text{in}}} \exp[-\mathrm{i}\omega_0 t] + \hat{a}_r^{\dagger\substack{\text{out}\\\text{in}}} \exp[+\mathrm{i}\omega_0 t], \tag{1.2.30q}$$

$$\begin{aligned}\hat{p}_r(t) &= m\frac{\mathrm{d}}{\mathrm{d}t}\hat{q}_r(t)\\ &\to (-\mathrm{i}m\omega_0)\left(\hat{a}_r^{\substack{\text{out}\\\text{in}}} \exp[-\mathrm{i}\omega_0 t] - \hat{a}_r^{\dagger\substack{\text{out}\\\text{in}}} \exp[+\mathrm{i}\omega_0 t]\right).\end{aligned} \tag{1.2.30p}$$

Thus, we obtain $\hat{a}_r^{\substack{\text{out}\\\text{in}}}$ as $t \to \pm\infty$,

$$\frac{\exp[\mathrm{i}\omega_0 t]}{2}\left(\frac{\mathrm{i}}{m\omega_0}\hat{p}_r(t) + \hat{q}_r(t)\right) \to \hat{a}_r^{\substack{\text{out}\\\text{in}}}, \qquad r = 1,\ldots,f\,. \tag{1.2.31}$$

We have the definition of the vacuum states, $|0,{}^{\text{out}}_{\text{in}}\rangle$, and their expansions in terms of $|q_{\pm\infty},\pm\infty\rangle$, as follows:

$$\begin{aligned}&\hat{a}_r^{\substack{\text{out}\\\text{in}}}|0,{}^{\text{out}}_{\text{in}}\rangle = 0,\\ &|0,{}^{\text{out}}_{\text{in}}\rangle = \int_{-\infty}^{+\infty}\mathrm{d}^f q_{\pm\infty}|q_{\pm\infty},\pm\infty\rangle\langle q_{\pm\infty},\pm\infty|0,{}^{\text{out}}_{\text{in}}\rangle\,.\end{aligned} \tag{1.2.32}$$

From (1.2.31) and (1.2.32), we obtain the following differential equation for the wave function of the vacuum,

$$\begin{aligned}&\hat{a}_r^{\substack{\text{out}\\\text{in}}}|0,{}^{\text{out}}_{\text{in}}\rangle\\ &\quad= \int_{-\infty}^{+\infty}\mathrm{d}^f q_{\pm\infty}\left(\frac{\mathrm{i}}{m\omega_0}\hat{p}_r(\pm\infty) + \hat{q}_r(\pm\infty)\right)|q_{\pm\infty},\pm\infty\rangle\\ &\qquad\times\langle q_{\pm\infty},\pm\infty|0,{}^{\text{out}}_{\text{in}}\rangle\\ &\quad= \int_{-\infty}^{+\infty}\mathrm{d}^f q_{\pm\infty}\left(-\frac{\hbar}{m\omega_0}\frac{\partial}{\partial q_r(\pm\infty)} + q_r(\pm\infty)\right)|q_{\pm\infty},\pm\infty\rangle\\ &\qquad\times\langle q_{\pm\infty},\pm\infty|0,{}^{\text{out}}_{\text{in}}\rangle\\ &\quad= \int_{-\infty}^{+\infty}\mathrm{d}^f q_{\pm\infty}|q_{\pm\infty},\pm\infty\rangle\left(\frac{\hbar}{m\omega_0}\frac{\partial}{\partial q_r(\pm\infty)} + q_r(\pm\infty)\right)\\ &\qquad\times\langle q_{\pm\infty},\pm\infty|0,{}^{\text{out}}_{\text{in}}\rangle\\ &\quad= 0,\end{aligned}$$

i.e.,

$$\left(\frac{\hbar}{m\omega_0}\frac{\partial}{\partial q_r(\pm\infty)}+q_r(\pm\infty)\right)\langle q_{\pm\infty},\pm\infty|0,{}^{\text{out}}_{\text{in}}\rangle=0, \quad r=1,\ldots,f\,. \tag{1.2.33}$$

We can solve (1.2.33) for $\langle q_{\pm\infty},\pm\infty|0,{}^{\text{out}}_{\text{in}}\rangle$ immediately, aside from the normalization constants, as

$$\langle q_{\pm\infty},\pm\infty|0,{}^{\text{out}}_{\text{in}}\rangle = \exp\left[-\left(\frac{m\omega_0}{2\hbar}\right)\sum_{r=1}^{f} q_r(\pm\infty)q_r(\pm\infty)\right]. \tag{1.2.34}$$

We note the following representations for $q_r(\pm\infty)$ with the use of the delta function at $t=\pm\infty$:

$$q_r(+\infty) = \frac{\varepsilon}{\hbar}\int_0^{+\infty} \mathrm{d}\tau \exp\left[-\frac{\varepsilon|\tau|}{\hbar}\right] q_r(\tau)\,, \tag{1.2.35a}$$

$$q_r(-\infty) = \frac{\varepsilon}{\hbar}\int_{-\infty}^{0} \mathrm{d}\tau \exp\left[-\frac{\varepsilon|\tau|}{\hbar}\right] q_r(\tau)\,, \tag{1.2.35b}$$

with

$$\varepsilon \to 0^+, \qquad r=1,\ldots,f.$$

We observe that the positive infinitesimal $\varepsilon$ has the dimension of energy. From (1.2.35a) and (1.2.35b), we obtain

$$\begin{aligned}&\langle q_{-\infty},-\infty|0,\text{in}\rangle\\ &\quad= \exp\left[-\left(\frac{m\omega_0}{2\hbar}\right)\frac{\varepsilon}{\hbar}\int_{-\infty}^{0} \mathrm{d}t \exp\left[-\frac{\varepsilon|t|}{\hbar}\right]\sum_{r=1}^{f} q_r^2(t)\right],\end{aligned} \tag{1.2.36a}$$

$$\begin{aligned}&\langle q_{+\infty},+\infty|0,\text{out}\rangle\\ &\quad= \exp\left[-\left(\frac{m\omega_0}{2\hbar}\right)\frac{\varepsilon}{\hbar}\int_{0}^{+\infty} \mathrm{d}t \exp\left[-\frac{\varepsilon|t|}{\hbar}\right]\sum_{r=1}^{f} q_r^2(t)\right],\end{aligned} \tag{1.2.36b}$$

$$\sum_{r=1}^{f}\left\{q_r^2(-\infty)+q_r^2(+\infty)\right\} = \frac{\varepsilon}{\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \exp\left[-\frac{\varepsilon|t|}{\hbar}\right]\sum_{r=1}^{f} q_r^2(t)\,, \tag{1.2.36c}$$

with

$$\varepsilon \to 0^+.$$

### 1.2.5 Generating Functional of the Green's Function

We define the generating functionals $Z[J]$ and $W[J]$ of the Green's function and its connected part (to be defined shortly) by

$$
\begin{aligned}
Z[J] &\equiv \exp\left[\frac{\mathrm{i}}{\hbar}W[J]\right] \\
&\equiv \left\langle 0,\mathrm{out}\left|\mathrm{T}\left(\exp\left[\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t\sum_{r=1}^{f}\hat{q}_r(t)J_r(t)\right]\right)\right|0,\mathrm{in}\right\rangle \\
&\equiv \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{i}}{\hbar}\right)^n\int_{-\infty}^{+\infty}\mathrm{d}t_1\cdots\mathrm{d}t_n\sum_{r_1,\dots,r_n=1}^{f}J_{r_1}(t_1)\cdots J_{r_n}(t_n) \\
&\quad\times\langle 0,\mathrm{out}|\mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n))|0,\mathrm{in}\rangle\,. \qquad (1.2.37)
\end{aligned}
$$

We have the following identities:

$$
\begin{aligned}
&\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_1}(t_1)}\cdots\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_n}(t_n)}Z[J]|_{J=0} \\
&\quad=\langle 0,\mathrm{out}|\mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n))|0,\mathrm{in}\rangle\,, \qquad (1.2.38\mathrm{Z})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_1}(t_1)}\cdots\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_{r_n}(t_n)}W[J]|_{J=0} \\
&\quad=\frac{\hbar}{\mathrm{i}}\langle 0,\mathrm{out}|\mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n))|0,\mathrm{in}\rangle_{\mathrm{C}}\,, \qquad (1.2.38\mathrm{W})
\end{aligned}
$$

where (1.2.38W) defines the connected part of the Green's function. In the definition of $Z[J]$, (1.2.37), we apply the resolution of identities at $t=\pm\infty$ in the $q$-representation and make use of the definition of $Z[J;q(t_b),q(t_a)]$, (1.2.28), and the expressions for $\langle q_{\pm\infty},\pm\infty|0,{}^{\mathrm{out}}_{\mathrm{in}}\rangle$, (1.2.36a, b), obtaining

$$
\begin{aligned}
Z[J] &= \left\langle 0,\mathrm{out}\left|\mathrm{T}\left(\exp\left[\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t\sum_{r=1}^{f}\hat{q}_r(t)J_r(t)\right]\right)\right|0,\mathrm{in}\right\rangle \\
&= \int_{-\infty}^{+\infty}\mathrm{d}^f q_{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}^f q_{-\infty}\langle 0,\mathrm{out}|q_{+\infty},+\infty\rangle \\
&\quad\times\left\langle q_{+\infty},+\infty\left|\mathrm{T}\left(\exp\left[\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t\sum_{r=1}^{f}\hat{q}_r(t)J_r(t)\right]\right)\right|q_{-\infty},-\infty\right\rangle \\
&\quad\times\langle q_{-\infty},-\infty|0,\mathrm{in}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \mathrm{d}^f q_{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}^f q_{-\infty} \int_{q(-\infty)=q_{-\infty}}^{q(+\infty)=q_{+\infty}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] \\
&\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \left\{\sum_{r=1}^{f} p_r(t)\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - H(q(t),p(t))\right.\right. \\
&\quad \left.\left. + \sum_{r=1}^{f} q_r(t)J_r(t) + \mathrm{i}\varepsilon\frac{m\omega_0}{2\hbar}\exp\left[-\frac{\varepsilon|t|}{\hbar}\right]\sum_{r=1}^{f} q_r^2(t)\right\}\right] \\
&= \int \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \left\{\sum_{r=1}^{f} p_r(t)\frac{\mathrm{d}}{\mathrm{d}t}q_r(t)\right.\right. \\
&\quad -H(q(t),p(t)) + \sum_{r=1}^{f} q_r(t)J_r(t) \\
&\quad \left.\left. +\mathrm{i}\varepsilon\frac{m\omega_0}{2\hbar}\exp\left[-\frac{\varepsilon|t|}{\hbar}\right]\sum_{r=1}^{f} q_r^2(t)\right\}\right] .
\end{aligned} \tag{1.2.39}
$$

We observe that the coordinate integration in the phase space path integral formula of $Z[J]$, (1.2.39), is no longer constrained as a result of the consideration of the vacuum wave functions, $\langle q_{\pm\infty}, \pm\infty | 0, {}^{\mathrm{out}}_{\mathrm{in}} \rangle$.

### 1.2.6 Configuration Space Path Integral Representation

We now perform the momentum integration in the phase space path integral formulas, (1.2.17), (1.2.20), (1.2.24) and (1.2.39), to obtain the configuration space path integral formula.

In general, Hamiltonian $H(q(t), p(t))$ is given by the quadratic form in the canonically conjugate momentum $p_r(t)$. We assume that the Hamiltonian $H(q(t), p(t))$ is of the form

$$
\begin{aligned}
H(q(t),p(t)) = &\sum_{r,s=1}^{f} \frac{1}{2} p_r(t) D_{r,s}(q(t)) p_s(t) \\
&+ \sum_{r=1}^{f} p_r(t) C_r(q(t)) + V(q(t)) ,
\end{aligned} \tag{1.2.40}
$$

where the kernel

$$\{D_{r,s}(q(t))\}_{r,s=1}^{f}$$

of the quadratic part is assumed to be real, symmetric and positive definite. With the Hamiltonian $H(q(t), p(t))$ of the form (1.2.40), we have

$$
\begin{aligned}
&\sum_{r=1}^{f} p_r(t)\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - H(q(t),p(t)) \\
&\qquad = -\sum_{r,s=1}^{f}\frac{1}{2}p_r(t)D_{r,s}(q(t))p_s(t) \\
&\qquad\qquad +\sum_{r=1}^{f}p_r(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - C_r(q(t))\right) - V(q(t))\,.
\end{aligned}
\tag{1.2.41}
$$

Thus we can perform the momentum integration in (1.2.17), (1.2.20), (1.2.24) and (1.2.39) as a quasi-Gaussian integral. We have

$$
\begin{aligned}
&\int \mathcal{D}[p(t)]\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}\mathrm{d}t\left\{\sum_{r=1}^{f}p_r(t)\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - H(q(t),p(t))\right\}\right] \\
&\qquad = [\mathrm{Det}D(q;t_b,t_a)]^{-1/2} \\
&\qquad\qquad \times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}\mathrm{d}t\left\{\begin{array}{l}\text{stationary value of} \\ \displaystyle\sum_{r=1}^{f}p_r(t)\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - H(q(t),p(t)) \\ \text{with respect to } p_r(t)\end{array}\right\}\right].
\end{aligned}
\tag{1.2.42}
$$

(For the finite-dimensional version of (1.2.42), we refer the reader to Appendix 1.) The stationary condition of (1.2.41) with respect to $p_r(t)$ is given by

$$
\begin{aligned}
&\frac{\partial}{\partial p_r(t)}\left\{\sum_{s=1}^{f}p_s(t)\frac{\mathrm{d}}{\mathrm{d}t}q_s(t) - H(q(t),p(t))\right\} \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - \{D_{r,s}(q(t))p_s(t) + C_r(q(t))\} \\
&\qquad = 0,
\end{aligned}
$$

i.e.,

$$
\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) = \frac{\partial H(q(t),p(t))}{\partial p_r(t)} = D_{r,s}(q(t))p_s(t) + C_r(q(t))\,. \tag{1.2.43}
$$

Since (1.2.43) is one of the canonical equations of motion, we obtain the canonical definition of $p_r(t)$ when (1.2.43) is solved for $p_r(t)$ as a function of $q_s(t)$ and $\mathrm{d}q_s(t)/\mathrm{d}t$:

$$
p_r(t) = [D^{-1}(q(t))]_{r,s}\left[\frac{\mathrm{d}}{\mathrm{d}t}q_s(t) - C_s(q(t))\right]. \tag{1.2.44a}
$$

From the definition of the Hamiltonian $H(q(t), p(t))$, (1.1.5), the stationary value of (1.2.41) with respect to $p_r(t)$ is the Lagrangian $L(q(t), \mathrm{d}q(t)/\mathrm{d}t)$, and is given by

$$
\begin{aligned}
&\sum_{r=1}^{f} p_r(t)\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) \\
&\qquad -H(q(t),p(t))\Big|_{p_r(t)=p_r(q(t),\,\mathrm{d}q(t)/\mathrm{d}t)\ \text{obtained from one of the canonical equations of motion, (1.2.43)}} \\
&\qquad \equiv L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) \\
&\qquad = \sum_{r,s=1}^{f} \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - C_r(q(t))\right][D^{-1}(q(t))]_{r,s} \\
&\qquad\quad \times \left[\frac{\mathrm{d}}{\mathrm{d}t}q_s(t) - C_s(q(t))\right] - V(q(t))\,.
\end{aligned}
\tag{1.2.45}
$$

From the Lagrangian $L(q(t), \mathrm{d}q(t)/\mathrm{d}t)$, (1.2.45), we observe that (1.2.44a) agrees with the definition of the momentum $p_r(t)$ canonically conjugate to $q_r(t)$,

$$
p_r(t) \equiv \frac{\partial L(q(t),\, \mathrm{d}q(t)/\mathrm{d}t)}{\partial\left(\mathrm{d}q_r(t)/\mathrm{d}t\right)}, \tag{1.2.44b}
$$

and that $\mathrm{d}q_r(t)/\mathrm{d}t$ becomes a function of $q_s(t)$ and $p_s(t)$ as the result of the momentum integration. Hence, we obtain the configuration space path integral formula:

**Transformation Function**:

$$
\begin{aligned}
&\langle q_{t_b}, t_b | q_{t_a}, t_a\rangle \\
&\qquad = \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)](\mathrm{Det}D(q;t_b,t_a))^{-1/2} \\
&\qquad\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)\right].
\end{aligned}
\tag{1.2.46}
$$

**Matrix Element of the Time-Ordered Product**:

$$
\begin{aligned}
&\langle q_{t_b}, t_b | \mathrm{T}(\hat{q}_{r_1}(t_1)\cdots\hat{q}_{r_n}(t_n)) | q_{t_a}, t_a\rangle \\
&\qquad = \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)](\mathrm{Det}D(q;t_b,t_a))^{-1/2} q_{r_1}(t_1)\cdots q_{r_n}(t_n) \\
&\qquad\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)\right].
\end{aligned}
\tag{1.2.47}
$$

**Matrix Element of the Operator $O(q(t))$:**

$$
\begin{aligned}
&\langle q_{t_b}, t_b | O(\hat{q}(t)) | q_{t_a}, t_a \rangle \\
&\quad = \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] (\mathrm{Det} D(q; t_b, t_a))^{-1/2} O(q(t)) \\
&\qquad \times \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{t_a}^{t_b} \mathrm{d}t L \left( q(t), \frac{\mathrm{d}}{\mathrm{d}t} q(t) \right) \right] .
\end{aligned}
\tag{1.2.48}
$$

**Generating Functional of the Green's Function**:

$$
\begin{aligned}
Z[J] &= \exp\left[ \frac{\mathrm{i}}{\hbar} W[J] \right] \\
&= \left\langle 0, \mathrm{out} \left| \mathrm{T} \left( \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{-\infty}^{+\infty} \mathrm{d}t \sum_{r=1}^{f} \hat{q}_r(t) J_r(t) \right] \right) \right| 0, \mathrm{in} \right\rangle \\
&= \int \mathcal{D}[q(t)] (\mathrm{Det} D(q))^{-1/2} \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{-\infty}^{+\infty} \mathrm{d}t \left\{ L \left( q(t), \frac{\mathrm{d}}{\mathrm{d}t} q(t) \right) \right.\right. \\
&\qquad \left.\left. + \sum_{r=1}^{f} q_r(t) J_r(t) + \text{"i}\varepsilon\text{-piece"} \right\} \right] .
\end{aligned}
\tag{1.2.49}
$$

When the kernel $\{D_{r,s}(q(t))\}_{r,s=1}^{f}$ of the quadratic form is a constant matrix, we find that the Feynman path integral formula derived in Sect. 1.1 is correct.

When the kernel of the quadratic form is a $q$-dependent matrix, we observe the following:

$$
\mathrm{Det} D(q; t_b, t_a) = \lim_{\substack{n \to \infty \\ t_0 = t_a \\ t_n = t_b}} \prod_{k=0}^{n-1} \det D(q(t_k)) ,
\tag{1.2.50a}
$$

$$
\mathrm{Det} D(q(t)) = \lim_{\substack{t_a \to -\infty \\ t_b \to +\infty}} \mathrm{Det} D(q; t_b, t_a) .
\tag{1.2.50b}
$$

We recall the formula for a real, symmetric and positive definite matrix $M$:

$$
\det M = \exp[\mathrm{tr} \ln M] .
\tag{1.2.51}
$$

Applying (1.2.51) to (1.2.50a) and (1.2.50b), we obtain

$$\prod_{k=0}^{n-1} \det D(q(t_k)) = \exp\left[\sum_{k=0}^{n-1} \mathrm{tr}\ln D(q(t_k))\right]$$
$$= \exp\left[\frac{1}{\delta t}\sum_{k=0}^{n-1} \delta t \mathrm{tr}\ln D(q(t_k))\right].$$

In general, in the limit as $n \to \infty$, we have

$$\frac{1}{\delta t} \to \delta(0), \qquad \sum_{k=0}^{n-1} \delta t \cdot f(t_k) \to \int_{t_a}^{t_b} \mathrm{d}t f(t),$$

so that

$$\mathrm{Det}D(q; t_b, t_a) = \exp\left[\delta(0)\int_{t_a}^{t_b} \mathrm{d}t \mathrm{tr}\ln D(q(t); t_b, t_a)\right], \tag{1.2.52a}$$

$$\mathrm{Det}D(q(t)) = \exp\left[\delta(0)\int_{-\infty}^{+\infty} \mathrm{d}t \mathrm{tr}\ln D(q(t))\right]. \tag{1.2.52b}$$

Hence, when the kernel of the quadratic form is the $q$-dependent matrix, from (1.2.52a) and (1.2.52b), we have in the integrand of (1.2.46) through (1.2.49),

$$(\mathrm{Det}D(q; t_b, t_a))^{-1/2} \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)\right]$$
$$= \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t \left\{L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) - \mathrm{i}\frac{\hbar}{2}\delta(0)\mathrm{tr}\ln D^{-1}(q(t); t_b, t_a)\right\}\right], \tag{1.2.53a}$$

$$(\mathrm{Det}D(q(t)))^{-1/2} \exp\left[\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)\right]$$
$$= \exp\left[\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \left\{L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) - \mathrm{i}\frac{\hbar}{2}\delta(0)\mathrm{tr}\ln D^{-1}(q(t))\right\}\right]. \tag{1.2.53b}$$

Thus, we find from (1.2.53a) and (1.2.53b) that we will get the correct result if we replace the original Lagrangian,

$$L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right),$$

by the effective Lagrangian,

$$L_{\text{eff}}\left(q(t),\frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)\equiv L\left(q(t),\frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)-\mathrm{i}\frac{\hbar}{2}\delta(0)\operatorname{tr}\ln D^{-1}(q(t))\,,\quad(1.2.54)$$

in the Feynman path integral formula of Sect. 1.1. We note that the matrix $D^{-1}(q(t))$ is the $q$-dependent "mass" matrix $M(q(t))$, as inspection of (1.2.40) and (1.2.45) implies,

$$D^{-1}(q(t))=M(q(t))\,.\quad(1.2.55)$$

In all of the discussions so far, ultimately leading to (1.2.54), we have assumed that the Hamiltonian $H(\hat{q}(t),\hat{p}(t))$ is a "well-ordered" operator. It is about time to address ourselves to the problem of operator ordering.

## 1.3 Weyl Correspondence

In the discussions so far, we have utterly evaded the problem of operator ordering with the notion of a "well-ordered" operator in order to provide the essence of path integral quantization. In this section, we shall discuss the problem of operator ordering squarely with the notion of the Weyl correspondence.

In Sect. 1.3.1, with the notion of the Weyl correspondence, we discuss the correspondence of analytical mechanics and quantum mechanics. In Sect. 1.3.2, using the result of the previous section, we reconsider the path integral formula in the Cartesian coordinate system and derive the mid-point rule as a natural consequence of the Weyl correspondence. In Sect. 1.3.3, we discuss the path integral formula in a curvilinear coordinate system. We first perform a coordinate transformation on the classical Hamiltonian and diagonal quantum Hamiltonian in the Cartesian coordinate system to the curvilinear coordinate system. Next, from the interpretation of the wave function as the probability amplitude in the curvilinear coordinate system, we determine the quantum Hamiltonian in the curvilinear coordinate system. From the comparison of the inverse Weyl transform of the classical Hamiltonian in the curvilinear coordinate system with the previously determined quantum Hamiltonian in the curvilinear coordinate system, we find the new effective potential $V_{\mathrm{c}}(\hat{q}(t))$ in the quantum Hamiltonian, which originates from the Jacobian of the coordinate transformation. In this way, we see that the Weyl transform of the quantum Hamiltonian in the curvilinear coordinate system is given by the sum of the classical Hamiltonian in the curvilinear coordinate system and the new effective potential $V_{\mathrm{c}}(q(t))$. We perform the momentum integration at this stage, and obtain the configuration space path integral formula in the curvilinear coordinate system.

From these considerations, we conclude that the *"new" effective Lagrangian*,

$$L_{\text{eff}}^{\text{new}}\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) \equiv L_{\text{eff}}\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) - V_{\mathrm{c}}(q(t)),$$

should be used in the Feynman path integral formula in Sects. 1.1 and 1.2 instead of the *"old" effective Lagrangian*,

$$L_{\text{eff}}\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) = L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) - \mathrm{i}\frac{\hbar}{2}\delta(0)\mathrm{tr}\ln D^{-1}(q(t)),$$

where

$$L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)$$

is the *original Lagrangian*. We note that the matrix $D^{-1}(q(t))$ is the $q$-dependent "mass" matrix $M(q(t))$ in this section,

$$D^{-1}(q(t)) = M(q(t)).$$

### 1.3.1 Weyl Correspondence

For an arbitrary operator $A(\hat{q}, \hat{p})$, we have the identity

$$A(\hat{q}, \hat{p}) = \iiiint \mathrm{d}^f p' \mathrm{d}^f p'' \mathrm{d}^f q' \mathrm{d}^f q'' |p''\rangle\langle p''|q''\rangle\langle q''|A(\hat{q}, \hat{p})|q'\rangle \times \langle q'|p'\rangle\langle p'| \,. \tag{1.3.1}$$

On the right-hand side of (1.3.1), we perform the change of the variables whose Jacobian is unity,

$$\begin{cases} q' = q - \frac{v}{2}, \\ p' = p - \frac{u}{2}, \end{cases} \qquad \begin{cases} q'' = q + \frac{v}{2}, \\ p'' = p + \frac{u}{2}, \end{cases} \qquad \begin{cases} q'' - q' = v, \\ p'' - p' = u. \end{cases} \tag{1.3.2}$$

Making use of the identity (1.2.10),

$$\langle q, t|p, t\rangle = \langle q|p\rangle = (2\pi\hbar)^{-f/2} \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{r=1}^{f} p_r q_r\right], \tag{1.2.10}$$

we can rewrite (1.3.1) as

$$\begin{aligned} A(\hat{q}, \hat{p}) = \iiiint & \frac{\mathrm{d}^f q \mathrm{d}^f p}{(2\pi\hbar)^f} \mathrm{d}^f u \mathrm{d}^f v \left|p + \frac{u}{2}\right\rangle \\ & \times \left\langle q + \frac{v}{2}\right| A(\hat{q}, \hat{p}) \left|q - \frac{v}{2}\right\rangle \left\langle p - \frac{u}{2}\right| \\ & \times \exp\left[-\frac{\mathrm{i}}{\hbar}\sum_{r=1}^{f}(p_r v_r + q_r u_r)\right]. \end{aligned} \tag{1.3.3}$$

We define the Weyl transform $a(q,p)$ of the operator $A(\hat{q},\hat{p})$ by

$$a(q,p) \equiv \int \mathrm{d}^f v \exp\left[-\frac{\mathrm{i}}{\hbar}\sum_{r=1}^{f} p_r v_r\right] \left\langle q + \frac{v}{2} | A(\hat{q},\hat{p}) | q - \frac{v}{2} \right\rangle , \tag{1.3.4}$$

and the inverse Weyl transform of the $c$-number function $a(q,p)$ by

$$A(\hat{q},\hat{p}) \equiv \iint \frac{\mathrm{d}^f q \mathrm{d}^f p}{(2\pi\hbar)^f} a(q,p) \Delta(q,p) , \tag{1.3.5}$$

where the Hermitian operator $\Delta(q,p)$ is defined by

$$\Delta(q,p) \equiv \int \mathrm{d}^f u \exp\left[-\frac{\mathrm{i}}{\hbar}\sum_{r=1}^{f} q_r u_r\right] \left| p + \frac{u}{2} \right\rangle \left\langle p - \frac{u}{2} \right| . \tag{1.3.6}$$

The matrix element of the operator $\Delta(q,p)$ in the $q$-representation is given by

$$\langle q_b | \Delta(q,p) | q_a \rangle = \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{r=1}^{f} p_r (q_b - q_a)_r\right] \delta^f \left( q - \frac{q_a + q_b}{2} \right) . \tag{1.3.7}$$

The delta function in (1.3.7) is the origin of the *mid-point rule.*

We record here the properties of the Weyl correspondence:

(a) The Weyl correspondence is one to one,

$$A(\hat{q},\hat{p}) \leftrightarrow a(q,p) . \tag{1.3.8a}$$

(b) When the Weyl correspondence, (1.3.8a), holds, we have the following Weyl correspondence for the Hermitian conjugate,

$$A^\dagger(\hat{q},\hat{p}) \leftrightarrow a^*(q,p) . \tag{1.3.8b}$$

(c) If $A(\hat{q},\hat{p})$ is Hermitian, $a(q,p)$ is real. Conversely, if $a(q,p)$ is real, then $A(\hat{q},\hat{p})$ is Hermitian.

We list several examples of the Weyl correspondence:

(1) $$(q^m p)_{\mathrm{W}} = \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} \hat{q}^{m-l} \hat{p} \hat{q}^l = \frac{1}{2}(\hat{q}^m \hat{p} + \hat{p}\hat{q}^m),$$

(2) $$(q^m p^2)_{\mathrm{W}} = \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} \hat{q}^{m-l} \hat{p}^2 \hat{q}^l = \frac{1}{4}(\hat{q}^m \hat{p}^2 + 2\hat{p}\hat{q}^m\hat{p} + \hat{p}^2\hat{q}^m),$$

(3) $(f(q)p)_{\mathrm{W}} = \frac{1}{2}(f(\hat{q})\hat{p} + \hat{p}f(\hat{q})),$

(4) $(f(q)p^2)_{\mathrm{W}} = \frac{1}{4}(f(\hat{q})\hat{p}^2 + 2\hat{p}f(\hat{q})\hat{p} + \hat{p}^2 f(\hat{q})).$

Examples (3) and (4) follow from examples (1) and (2) by the power series expansion of $f(q)$. We will use some of these examples shortly.

### 1.3.2 Path Integral Formula in a Cartesian Coordinate System

When we have the classical Hamiltonian $H_{\mathrm{cl}}(q,p)$ in the Cartesian coordinate system, we define the quantum Hamiltonian $H_{\mathrm{q}}(\hat{q},\hat{p})$ as the inverse Weyl transform of $H_{\mathrm{cl}}(q,p)$,

$$H_{\mathrm{q}}(\hat{q},\hat{p}) \leftrightarrow H_{\mathrm{cl}}(q,p).$$

$$H_{\mathrm{q}}(\hat{q},\hat{p}) \equiv \iint \frac{\mathrm{d}^f q \mathrm{d}^f p}{(2\pi\hbar)^f} H_{\mathrm{cl}}(q,p) \Delta(q,p)\,. \tag{1.3.9}$$

We divide the time interval $[t_a, t_b]$ into $n$ equal subintervals, $[t_k, t_{k+1}]$,

$$\delta t = \frac{t_b - t_a}{n}, \quad t_k = t_a + k\delta t, \quad k = 0, 1, \ldots, n-1, n; \quad t_n \equiv t_b, \quad t_0 \equiv t_a,$$

and obtain the transformation function $\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle$,

$$\begin{aligned} &\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle \\ &\quad = \int \cdots \int \prod_{j=1}^{n-1} \mathrm{d}^f q_{t_j} \prod_{j=0}^{n-1} \langle q_{t_{j+1}}, t_{j+1} | q_{t_j}, t_j \rangle \\ &\quad = \int \cdots \int \prod_{j=1}^{n-1} \mathrm{d}^f q_{t_j} \prod_{j=0}^{n-1} \left\langle q_{t_{j+1}} \middle| \exp\left[-\frac{\mathrm{i}}{\hbar}\delta t H_{\mathrm{q}}(\hat{q},\hat{p})\right] \middle| q_{t_j} \right\rangle. \end{aligned} \tag{1.3.10}$$

According to the Weyl correspondence, (1.3.9), we have the following Weyl correspondence to first order in $\delta t$,

$$\exp\left[-\frac{\mathrm{i}}{\hbar}\delta t H_{\mathrm{q}}(\hat{q},\hat{p})\right] \leftrightarrow \exp\left[-\frac{\mathrm{i}}{\hbar}\delta t H_{\mathrm{cl}}(q,p)\right]. \tag{1.3.11}$$

Hence, to first order in $\delta t$, we have the identity,

$$\begin{aligned}
&\left\langle q_{t_{j+1}} \left| \exp\left[-\frac{\mathrm{i}}{\hbar}\delta t H_{\mathrm{q}}(\hat{q},\hat{p})\right] \right| q_{t_j} \right\rangle \\
&\quad = \iint \frac{\mathrm{d}^f q \mathrm{d}^f p}{(2\pi\hbar)^f} \exp\left[-\frac{\mathrm{i}}{\hbar}\delta t H_{\mathrm{cl}}(q,p)\right] \langle q_{t_{j+1}} | \Delta(q,p) | q_{t_j} \rangle \\
&\quad = \int \frac{\mathrm{d}^f p}{(2\pi\hbar)^f} \exp\left[\frac{\mathrm{i}}{\hbar}\delta t \left\{ \sum_{r=1}^{f} \left( \frac{q_{t_j+\delta t} - q_{t_j}}{\delta t} \right)_r p_r \right.\right. \\
&\qquad\qquad \left.\left. - H_{\mathrm{cl}}\left( \frac{q_{t_{j+\delta t}} + q_{t_j}}{2}, p \right) \right\} \right] .
\end{aligned} \tag{1.3.12}$$

We note that the mid-point rule comes out as a natural consequence of the Weyl correspondence as indicated in (1.3.7) and (1.3.12). Substituting (1.3.12) into (1.3.10), we obtain to first order in $\delta t$,

$$\begin{aligned}
&\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle \\
&\quad = \iint \prod_{j=1}^{n-1} \mathrm{d}^f q_{t_j} \prod_{j=0}^{n-1} \frac{\mathrm{d}^f p_{t_j}}{(2\pi\hbar)^f} \\
&\qquad \times \exp\left[ \frac{\mathrm{i}}{\hbar} \sum_{j=0}^{n-1} \delta t \left\{ \sum_{r=1}^{f} \left( \frac{q_{t_j+\delta t} - q_{t_j}}{\delta t} \right)_r p_{t_j,r} \right.\right. \\
&\qquad\qquad \left.\left. - H_{\mathrm{cl}}\left( \frac{q_{t_j+\delta t} + q_{t_j}}{2}, p_{t_j} \right) \right\} \right] \\
&\quad = \int \frac{\mathrm{d}^f p_{t_a}}{(2\pi\hbar)^f} \iint \prod_{j=1}^{n-1} \frac{\mathrm{d}^f q_{t_j} \mathrm{d}^f p_{t_j}}{(2\pi\hbar)^f} \\
&\qquad \times \exp\left[ \frac{\mathrm{i}}{\hbar} \sum_{j=0}^{n-1} \delta t \left\{ \sum_{r=1}^{f} \left( \frac{q_{t_j+\delta t} - q_{t_j}}{\delta t} \right)_r p_{t_j,r} \right.\right. \\
&\qquad\qquad \left.\left. - H_{\mathrm{cl}}\left( \frac{q_{t_j+\delta t} + q_{t_j}}{2}, p_{t_j} \right) \right\} \right] .
\end{aligned} \tag{1.3.13}$$

We now take the limit $n \to \infty$, obtaining

$$
\begin{aligned}
&\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle \\
&\quad = \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \int \mathcal{D}[p(t)] \\
&\qquad \times \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{t_a}^{t_b} \mathrm{d}t \left\{ \sum_{r=1}^{f} p_r(t) \frac{\mathrm{d}}{\mathrm{d}t} q_r(t) - H_{\mathrm{cl}}(q(t), p(t)) \right\} \right]. \qquad (1.3.14)
\end{aligned}
$$

Here, the functional integral measure is given by

$$
\mathcal{D}[q(t)]\mathcal{D}[p(t)] \equiv \lim_{n\to\infty} \frac{\mathrm{d}^f p_{t_a}}{(2\pi\hbar)^f} \prod_{j=1}^{n-1} \frac{\mathrm{d}^f q_{t_j} \mathrm{d}^f p_{t_j}}{(2\pi\hbar)^f}\,. \qquad (1.3.15)
$$

At first sight, (1.3.14) looks quite elegant, but its meaning is nothing but the limit $n \to \infty$ of (1.3.13).

As pointed out earlier, we derived the mid-point rule not as an *ad hoc* approximation formula but as a natural consequence of the Weyl correspondence.

### 1.3.3 Path Integral Formula in a Curvilinear Coordinate System

In this subsection, in order to make a clear distinction between the Cartesian coordinate system and a curvilinear coordinate system, we use $x = \{x_r\}_{r=1}^{f}$ for the former and $q = \{q_r\}_{r=1}^{f}$ for the latter. We transform the diagonal classical Lagrangian

$$
L\left(x(t), \frac{\mathrm{d}}{\mathrm{d}t} x(t)\right) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} x^{\mathrm{T}}(t) \frac{\mathrm{d}}{\mathrm{d}t} x(t) - V(x(t)) \qquad (1.3.16)
$$

in the Cartesian coordinate system into the classical Lagrangian

$$
L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t} q(t)\right) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} q^{\mathrm{T}}(t) M(q(t)) \frac{\mathrm{d}}{\mathrm{d}t} q(t) - V(q(t)) \qquad (1.3.17)
$$

in the curvilinear coordinate system. Here, we have the Jacobian of the coordinate transformation $\mathcal{J}(q)$, the "mass" matrix $M(q(t))$, and the "inverse mass" matrix $M^{-1}(q(t))$ as

$$
\mathcal{J}(q) = \left| \det \left( \frac{\partial x_s(t)}{\partial q_r(t)} \right) \right|, \qquad (1.3.18)
$$

$$
M(q(t))_{r,s} = \sum_{k=1}^{f} \frac{\partial x_k(t)}{\partial q_r(t)} \frac{\partial x_k(t)}{\partial q_s(t)}\,, \qquad (1.3.19)
$$

and

$$M^{-1}(q(t))^{r,s} = \sum_{k=1}^{f} \frac{\partial q_r(t)}{\partial x_k(t)} \frac{\partial q_s(t)}{\partial x_k(t)} \,. \tag{1.3.20}$$

The "mass" matrix $M(q(t))$ corresponds to $D^{-1}(q(t))$ of Sect. 1.2.6. We define the matrix $\{J_{r,s}\}_{r,s=1}^{f}$ by

$$J_{r,s} \equiv \frac{\partial x_s(t)}{\partial q_r(t)}, \qquad r,s = 1,\ldots,f \,. \tag{1.3.21}$$

Then, we have

$$\mathcal{J}(q) = |\det(J_{r,s})|, \qquad M(q(t))_{r,s} = J_{r,k}J_{s,k} = (JJ^{\mathrm{T}})_{r,s} \,, \tag{1.3.22}$$

and hence we have

$$\mathcal{J}(q) = (\det M(q(t)))^{1/2} \,. \tag{1.3.23}$$

The momentum $p_r(t)$ canonically conjugate to the curvilinear coordinate $q_r(t)$ is defined by

$$p_r(t) \equiv \frac{\partial L\left(q(t), \mathrm{d}q(t)/\mathrm{d}t\right)}{\partial(\mathrm{d}q_r(t)/\mathrm{d}t)} = M(q(t))_{r,s} \frac{\mathrm{d}}{\mathrm{d}t} q_s(t) \,. \tag{1.3.24}$$

Solving (1.3.24) for $\mathrm{d}q_r(t)/\mathrm{d}t$, we have the classical Hamiltonian $H_{\mathrm{cl}}(q,p)$ in the curvilinear coordinate system as

$$\begin{aligned} H_{\mathrm{cl}}(q,p) &= \sum_{r=1}^{f} p_r(t)\frac{\mathrm{d}}{\mathrm{d}t}q_r(t) - L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) \\ &= \frac{1}{2}p^{\mathrm{T}}(t)M^{-1}(q(t))p(t) + V(q(t)) \,. \end{aligned} \tag{1.3.25}$$

As the inverse Weyl transform $H_{\mathrm{W}}(\hat{q},\hat{p})$ of the classical Hamiltonian $H_{\mathrm{cl}}(q,p)$, we have

$$\begin{aligned} H_{\mathrm{W}}(\hat{q},\hat{p}) = \frac{1}{8}\big[&\hat{p}_r(t)\hat{p}_s M^{-1}(\hat{q}(t))^{r,s} + 2\hat{p}_r M^{-1}(\hat{q}(t))^{r,s}\hat{p}_s(t) \\ &+ M^{-1}(\hat{q}(t))^{r,s}\hat{p}_r(t)\hat{p}_s(t)\big] + V(\hat{q}(t)) \,, \end{aligned} \tag{1.3.26}$$

where we have used Example (4) of Sect. 1.3.1. Next, we shall write the quantum Hamiltonian

$$H_{\mathrm{q}}\left(\hat{x}(t), \hat{p}(t) \equiv \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \hat{x}(t)}\right)$$

corresponding to the classical Lagrangian,

$$L\left(\hat{x}(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t)\right), \tag{1.3.16}$$

in the Cartesian coordinate system as

$$H_{\mathrm{q}}\left(\hat{x}(t), \hat{p}(t) \equiv \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \hat{x}(t)}\right) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial \hat{x}_i \partial \hat{x}_i} + V(\hat{x}(t))\,. \tag{1.3.27}$$

Performing the coordinate transformation to curvilinear coordinates on (1.3.27), we have the quantum Hamiltonian $H_{\mathrm{q}}(\hat{q}(t), \hat{p}(t))$ in the curvilinear coordinate system as

$$H_{\mathrm{q}}(\hat{q}(t), \hat{p}(t)) = \frac{1}{2}\frac{1}{\mathcal{J}(q)}\hat{p}_r(t)\left[M^{-1}(\hat{q}(t))^{r,s}\mathcal{J}(q)\hat{p}_s(t)\right] + V(\hat{q}(t))\,. \tag{1.3.28}$$

Here, we defined the momentum operator $\hat{p}_r(t)$ in the curvilinear coordinate system as

$$\hat{p}_r(t) \equiv \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \hat{q}_r(t)}, \qquad r = 1, \ldots, f\,, \tag{1.3.29a}$$

which is *Hermitian* with respect to the scalar product

$$\langle f_1 | f_2 \rangle = \int \cdots \int f_1^*(q) f_2(q) \prod_{r=1}^{f} \mathrm{d}q_r.$$

In deriving (1.3.28), we used the following formulas:

$$\text{(a)}\quad \frac{1}{\det M(q(t))}\frac{\partial}{\partial q_r(t)}\det M(q(t)) = M^{-1}(q(t))^{j,i}\frac{\partial}{\partial q_r(t)}M(q(t))_{i,j}$$
$$= -2\frac{\partial x_i(t)}{\partial q_r(t)}\frac{\partial}{\partial q_j(t)}\left(\frac{\partial q_j(t)}{\partial x_i(t)}\right), \tag{1.3.30a}$$

$$\text{(b)}\quad \frac{\partial^2}{\partial x_r(t)\partial x_r(t)} = \frac{1}{\mathcal{J}(q)}\frac{\partial}{\partial q_r(t)}\left[M^{-1}(q(t))^{r,s}\mathcal{J}(q)\frac{\partial}{\partial q_s(t)}\right]. \tag{1.3.30b}$$

Associated with the coordinate transformation, we know that the volume element $\prod_{r=1}^{f}\mathrm{d}x_r$ gets changed into

$$\prod_{r=1}^{f}\mathrm{d}x_r = \mathcal{J}(q)\prod_{r=1}^{f}\mathrm{d}q_r\,. \tag{1.3.31}$$

Hence, the wave function $\psi(x,t)$ in the Cartesian coordinate system is a scalar density with weight 1/2. We define the wave function $\phi(q,t)$ in the curvilinear coordinate system by

$$\psi(x,t) \equiv \langle x,t|\psi\rangle = \frac{1}{\sqrt{\mathcal{J}(q)}}\langle q,t|\psi\rangle \equiv \frac{1}{\sqrt{\mathcal{J}(q)}}\phi(q,t)\,. \tag{1.3.32}$$

With this choice of $\phi(q,t)$ as in (1.3.32), we have the interpretation of $|\phi(q,t)|^2$ as the probability density in the volume element $\prod_{r=1}^{f} \mathrm{d}q_r$ in the curvilinear coordinate system. We have the quantum Hamiltonian $\bar{H}_{\mathrm{q}}(\hat{q}(t),\hat{p}(t))$ of the wave function $\phi(q,t)$ from (1.3.32) as

$$\bar{H}_{\mathrm{q}}(\hat{q}(t),\hat{p}(t)) = \sqrt{\mathcal{J}(q)}H_{\mathrm{q}}(\hat{q}(t),\hat{p}(t))\frac{1}{\sqrt{\mathcal{J}(q)}}\,. \tag{1.3.33a}$$

We have the time-dependent Schrödinger equation satisfied by $\phi(q,t)$ as

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\phi(q,t) = \bar{H}_{\mathrm{q}}(\hat{q}(t),\hat{p}(t))\phi(q,t)\,. \tag{1.3.34}$$

Calculating $\bar{H}_{\mathrm{q}}(\hat{q}(t),\hat{p}(t))$ defined by (1.3.33a), with the use of (1.3.18), (1.3.28), (1.3.29a), (1.3.30a) and (1.3.30b), we finally obtain

$$\bar{H}_{\mathrm{q}}(\hat{q}(t),\hat{p}(t)) = H_{\mathrm{W}}(\hat{q}(t),\hat{p}(t)) + V_{\mathrm{c}}(\hat{q}(t))\,, \tag{1.3.35}$$

with

$$V_{\mathrm{c}}(q(t)) = \frac{\hbar^2}{8}\left[\frac{\partial}{\partial q_r(t)}\left(\frac{\partial q_s(t)}{\partial x_k(t)}\right)\right]\left[\frac{\partial}{\partial q_s(t)}\left(\frac{\partial q_r(t)}{\partial x_k(t)}\right)\right]. \tag{1.3.36a}$$

This $V_{\mathrm{c}}(q(t))$ is the *new effective potential* originating from the Jacobian of the coordinate transformation and the interpretation of the wave function $\phi(q,t)$ in the curvilinear coordinate system as the probability amplitude. Under the Weyl transformation, we have

$$V_{\mathrm{c}}(\hat{q}(t)) \leftrightarrow V_{\mathrm{c}}(q(t))\,, \tag{1.3.37}$$

so that for the quantum Hamiltonian $\bar{H}_{\mathrm{q}}(\hat{q}(t),\hat{p}(t))$ of the wave function $\phi(q,t)$, we have the following Weyl correspondence,

$$\bar{H}_{\mathrm{q}}(\hat{q}(t),\hat{p}(t)) = H_{\mathrm{W}}(\hat{q}(t),\hat{p}(t)) + V_{\mathrm{c}}(\hat{q}(t)) \leftrightarrow H_{\mathrm{cl}}(q(t),p(t)) + V_{\mathrm{c}}(q(t))$$

$$\bar{H}_{\mathrm{q}}(\hat{q}(t),\hat{p}(t)) \equiv \iint \frac{\mathrm{d}^f q\,\mathrm{d}^f p}{(2\pi\hbar)^f}\{H_{\mathrm{cl}}(q(t),p(t)) + V_{\mathrm{c}}(q(t))\}\varDelta(q(t),p(t))\,. \tag{1.3.38}$$

We may define the momentum operator $\hat{p}(t)$ in the curvilinear coordinate system as

$$\hat{p}_r(t) \equiv \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial q_r(t)} + \frac{1}{2}\varGamma_r\right), \quad \varGamma_r \equiv \frac{\partial}{\partial q_r}\ln\mathcal{J}(q), \quad r = 1,\ldots,f\,, \tag{1.3.29b}$$

which is *Hermitian* with respect to the scalar product

$$\langle f_1|f_2\rangle = \int \cdots \int f_1^*(q) f_2(q) \mathcal{J}(q) \prod_{r=1}^{f} \mathrm{d}q_r .$$

The quantum Hamiltonian $\bar{H}_\mathrm{q}(\hat{q}(t), \hat{p}(t))$ in the curvilinear coordinate system for the choice of the momentum operator $\hat{p}(t)$ as in (1.3.29b) is given by

$$\bar{H}_\mathrm{q}(\hat{q}(t), \hat{p}(t)) = \frac{1}{2} \frac{1}{\sqrt{\mathcal{J}(q)}} \hat{p}_r(t) [M^{-1}(q(t))^{r,s} \mathcal{J}(q) \hat{p}_s(t)] \frac{1}{\mathcal{J}(q)} + V(\hat{q}(t)) . \tag{1.3.33b}$$

We obtain the *new effective potential* $V_\mathrm{c}(q(t))$ as

$$V_\mathrm{c}(q(t)) = \frac{\hbar^2}{8} \left( M^{-1}(q(t))^{r,s} \Gamma^u_{rv} \Gamma^v_{su} - R \right) , \tag{1.3.36b}$$

where $R$ is the *scalar curvature*, and $\Gamma^\mathrm{i}_{jk}$ is the *Christoffel symbol* respectively given by

$$R = M^{-1}(q(t))^{r,s} \left( \frac{\partial \Gamma^l_{ls}}{\partial q_r} - \frac{\partial \Gamma^l_{rs}}{\partial q_l} + \Gamma^l_{ms} \Gamma^m_{rl} - \Gamma^l_{rs} \Gamma^m_{ml} \right) , \tag{1.3.36c}$$

and

$$\Gamma^\mathrm{i}_{jk} = \frac{1}{2} M^{-1}(q(t))^{i,a} \times \left( \frac{\partial M(q(t))_{j,a}}{\partial q_k} + \frac{\partial M(q(t))_{k,a}}{\partial q_j} - \frac{\partial M(q(t))_{j,k}}{\partial q_a} \right) . \tag{1.3.36d}$$

The choice of the momentum operator as in (1.3.29a) is non-standard but the expression for the effective potential $V_\mathrm{c}(q(t))$ as in (1.3.36a) is simple, while the choice of the momentum operator as in (1.3.29b) is standard but the expression for the effective potential $V_\mathrm{c}(q(t))$ as in (1.3.36b) is not simple.

As for the transformation function $\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle$, we divide the time interval $[t_a, t_b]$ into $n$ equal subintervals $[t_k, t_{k+1}]$,

$$\delta t = \frac{t_b - t_a}{n}, \quad t_k = t_a + k\delta t, \quad k = 0, 1, \ldots, n-1, n; \quad t_0 \equiv t_a, \quad t_n \equiv t_b,$$

and with an argument similar to that leading to (1.3.13), we have, to first order in $\delta t$,

$$\begin{aligned} &\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle \\ &\quad = \int \cdots \int \prod_{j=1}^{n-1} \mathrm{d}^f q_{t_j} \prod_{j=0}^{n-1} \langle q_{t_{j+1}}, t_{j+1} | q_{t_j}, t_j \rangle \\ &\quad = \int \cdots \int \prod_{j=1}^{n-1} \mathrm{d}^f q_{t_j} \prod_{j=0}^{n-1} \left\langle q_{t_{j+1}} \middle| \exp\left[ -\frac{\mathrm{i}}{\hbar} \delta t \bar{H}_\mathrm{q}(\hat{q}, \hat{p}) \right] \middle| q_{t_j} \right\rangle \end{aligned} \tag{1.3.39}$$

$$
\begin{aligned}
&= \int \frac{\mathrm{d}^f p_{t_a}}{(2\pi\hbar)^f} \int \cdots \int \prod_{j=1}^{n-1} \frac{\mathrm{d}^f q_{t_j} \mathrm{d}^f p_{t_j}}{(2\pi\hbar)^f} \\
&\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\delta t \left\{ \sum_{r=1}^{f} \left(\frac{q_{t_j+\delta t} - q_{t_j}}{\delta t}\right)_r p_{t_j,r} \right.\right. \\
&\quad \left.\left. - H_{\mathrm{cl}}\left(\frac{q_{t_j+\delta t} + q_{t_j}}{2}, p_{t_j}\right) - V_{\mathrm{c}}\left(\frac{q_{t_j+\delta t} + q_{t_j}}{2}\right)\right\}\right]. \qquad (1.3.40)
\end{aligned}
$$

We remark that (1.3.39) is exact, while (1.3.40) is correct to first order in $\delta t$. Recalling the fact that the classical Hamiltonian $H_{\mathrm{cl}}(q(t), p(t))$ is quadratic in $p(t)$, (1.3.25), we can perform the momentum integration as a quasi-Gaussian integral in (1.3.40) and can exponentiate the resulting determinant factor as in (1.2.51), obtaining

$$
\begin{aligned}
&\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle \\
&\quad = \int \prod_{j=1}^{n-1} \frac{\mathrm{d}^f q_{t_j}}{(2\pi\hbar)^f} \left(\frac{\hbar}{\mathrm{i}\delta t}\right)^{f/2} \\
&\qquad \times \exp\left[\frac{\mathrm{i}}{\hbar} \sum_{j=0}^{n-1} \delta t \cdot L_{\mathrm{eff}}^{\mathrm{new}}\left(\frac{q_{t_j} + q_{t_j+\delta t}}{2}, \frac{q_{t_j+\delta t} - q_{t_j}}{\delta t}\right)\right]. \qquad (1.3.41)
\end{aligned}
$$

We have the new effective Lagrangian in (1.3.41) as

$$
\begin{aligned}
&L_{\mathrm{eff}}^{\mathrm{new}}\left(\frac{q_{t_j} + q_{t_j+\delta t}}{2}, \frac{q_{t_j+\delta t} - q_{t_j}}{\delta t}\right) \\
&\quad = L\left(\frac{q_{t_j} + q_{t_j+\delta t}}{2}, \frac{q_{t_j+\delta t} - q_{t_j}}{\delta t}\right) \\
&\qquad - \mathrm{i}\frac{\hbar}{2}\frac{1}{\delta t} \mathrm{tr} \ln M\left(\frac{q_{t_j} + q_{t_j+\delta t}}{2}\right) - V_{\mathrm{c}}\left(\frac{q_{t_j} + q_{t_j+\delta t}}{2}\right). \qquad (1.3.42)
\end{aligned}
$$

Now, we take the limit as $n \to \infty$, obtaining

$$
\begin{aligned}
\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle &= \int_{q(t_a)=q_{t_a}}^{q(t_b)=q_{t_b}} \mathcal{D}[q(t)] \\
&\quad \times \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t_a}^{t_b} \mathrm{d}t L_{\mathrm{eff}}^{\mathrm{new}}\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t} q(t)\right)\right]. \qquad (1.3.43)
\end{aligned}
$$

The *new effective Lagrangian* $L_{\mathrm{eff}}^{\mathrm{new}}(q(t), \mathrm{d}q(t)/\mathrm{d}t)$ in (1.3.43) is the continuum version of (1.3.42), and is given by

$$L_{\text{eff}}^{\text{new}}\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right) = L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)$$

$$-\mathrm{i}\frac{\hbar}{2}\delta(0)\mathrm{tr}\ln M(q(t)) - V_{\mathrm{c}}(q(t)), \tag{1.3.44}$$

which is the new effective Lagrangian announced at the beginning of this section with the identification of $D^{-1}(q(t))$ with the "mass" matrix $M(q(t))$,

$$D^{-1}(q(t)) = M(q(t)). \tag{1.3.45}$$

We repeat that (1.3.43) and (1.3.44) are nothing but the limits as $n \to \infty$ of (1.3.41) and (1.3.42).

From these considerations, we conclude that, in order to write down the path integral formula in a curvilinear coordinate system, it is best to adopt the following procedure:

(1) Write down the path integral formula in the Cartesian coordinate system.
(2) Perform the coordinate transformation from the Cartesian coordinate system to the curvilinear coordinate system.
(3) Calculate the "mass" matrix $M(q(t))$ from (1.3.19) and the effective potential $V_{\mathrm{c}}(q(t))$ from (1.3.36a) or (1.3.36b).
(4) Determine the new effective Lagrangian $L_{\text{eff}}^{\text{new}}(q(t), \mathrm{d}q(t)/\mathrm{d}t)$ from (1.3.44) and the transformation function $\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle$ from (1.3.43).

## 1.4 Bibliography

We first list the standard text books on analytical mechanics, quantum mechanics, quantum field theory, quantum statistical mechanics and many body problem, and path integral quantization.

### Analytical Mechanics

(**T-1**) Goldstein, H.; "Classical Mechanics", Addison-Wesley Publishing Company, 1964, Massachusetts.
(**T-2**) Landau, L.D. and Lifshitz, E.M.; "Mechanics", 3rd edition, Pergamon Press Ltd., 1976, New York.
(**T-3**) Fetter, A.L. and Walecka, J.D.; "Theoretical Mechanics of Particles and Continua", McGraw-Hill, 1980, New York.

### Quantum Mechanics

(**T-4**) Dirac, P.A.M.; "Principles of Quantum Mechanics", 4th edition, Oxford Univ. Press, 1958, London.
(**T-5**) Schiff, L.I.; "Quantum Mechanics", 3rd edition, McGraw-Hill, 1968, New York.

(**T-6**) Gottfried, K.; ″Quantum Mechanics. I. Fundamentals″, Benjamin, 1966, New York.
(**T-7**) Landau, L.D. and Lifshitz, E.M.; ″Quantum Mechanics″, 3rd edition, Pergamon Press Ltd., 1977, New York.
(**T-8**) Sakurai, J.J.; ″Modern Quantum Mechanics″, Addison-Wesley Publishing Company, 1985, Massachusetts.

**Quantum Field Theory**

(**T-9**) Wentzel, G.; ″Quantum Theory of Fields″, Interscience Publishers Ltd., 1949, New York.
(**T-10**) Bogoliubov, N.N., and Shirkov, D.V.; ″Introduction to the Theory of Quantized Field″, 3rd edition, John Wiley and Sons, 1980, New York.
(**T-11**) Nishijima, K.; ″Fields and Particles: Field Theory and Dispersion Relations″, Benjamin Cummings, 1969 and 1974, Massachusetts.
(**T-12**) Itzykson, C. and Zuber, J.B.; ″Quantum Field Theory″, McGraw-Hill, 1980, New York.
(**T-13**) Lurie, D.; ″Particles and Fields″, John Wiley and Sons, 1968, New York.
(**T-14**) Weinberg, S.; ″Quantum Theory of Fields I″, Cambridge Univ. Press, 1995, New York.
(**T-15**) Weinberg, S.; ″Quantum Theory of Fields II″, Cambridge Univ. Press, 1996, New York.
(**T-16**) Weinberg, S.; ″Quantum Theory of Fields III″, Cambridge Univ. Press, 2000, New York.

**Quantum Statistical Mechanics and the Many Body Problem**

(**T-17**) Huang, K.; ″Statistical Mechanics″, 2nd edition, John Wiley and Sons, 1983, New York.
(**T-18**) Fetter, A.L. and Walecka, J.D.; ″Quantum Theory of Many-particle Systems″, McGraw-Hill, 1971, New York.

**Path Integral Quantization**

(**T-19**) Feynman, R.P. and Hibbs, A.R.; ″Quantum Mechanics and Path Integrals″, McGraw-Hill, 1965, New York.

Path integral quantization of *Non-Abelian gauge fields* is discussed in the following books, besides (**T-12**) and (**T-15**).

(**T-20**) Ramond, P.; ″Field Theory: A Modern Primer″, Benjamin Cummings, 1981, Massachusetts.
(**T-21**) Popov, V.N.; ″Functional Integrals in Quantum Field Theory and Statistical Physics″, Atomizdat, 1976, Moscow. English translation from Reidel, 1983, Boston.
(**T-22**) Vassiliev, A.N.; ″Functional Methods in Quantum Field Theory and Statistics″, Leningrad Univ. Press, 1976, Leningrad.

A mathematical discussion of path integral quantization is given in the following books.

(**T-23**) Simon, B.; ″Functional Integration and Quantum Physics″, Academic Press, 1979, New York.

(**T-24**) Glimm, J. and Jaffe, A.; ″Quantum Physics: Functional Integral Point of View″, Springer-Verlag, 1981, New York.

A variety of path integral techniques are discussed in the following book.

(**T-25**) Schulman, L.S.; ″Techniques and Application of Path Integration″, John Wiley and Sons, 1981, New York.

**Chapter 1. Section 1.1**

We cite the following two classic articles as the genesis of path integral quantization.

(**R1-1**) Dirac, P.A.M.; Physik. Z. Sowjetunion **3,** (1933), 64, reprinted in ″Selected Papers on Quantum Electrodynamics″, J. Schwinger, ed. Dover, 1958, New York.

(**R1-2**) Feynman, R.P.; Rev. Mod. Phys. **20**, (1948), 367, reprinted in ″Selected Papers on Quantum Electrodynamics″, J. Schwinger, ed. Dover, 1958, New York.

The concept of the probability amplitude associated with an arbitrary path in quantum mechanics is discussed in the following articles.

(**R1-3**) Feynman, R.P.; ″The Concept of Probability in Quantum Mechanics″, in Proc. of 2$^{\mathrm{nd}}$ Berkeley Symposium on Math. Stat. and Probability, 533.

(**R1-4**) Hibbs, A.R.; Appendix II of ″Probability and Related Topics in Physical Sciences″, Lect. in Applied Math., Proc. of Summer Seminar., Boulder, Colorado, 1957, Vol.1. Interscience Publishers Ltd., London.

**Chapter 1. Section 1.2**

We cite the appendices of the following article for the derivation of the configuration space path integral formula from the phase space path integral formula. The $q$-dependent kernel of the quadratic form in the canonical momentum $p$ is not discussed in this article.

(**R1-5**) Feynman, R.P.; Phys. Rev. **84**, (1951), 108; Mathematical Appendices A, B and C.

**Chapter 1. Section 1.3**

We cite the following articles for the resolution of the operator ordering problem with the notion of the Weyl correspondence.

(**R1-6**) Weyl, H.; Z. Physik. **46**, (1928) 1.

(**T-26**) Weyl, H.; ″Theory of Groups and Quantum Mechanics″, Leipzig, 1928, Zurich; reprinted by Dover, 1950, 275.

(**R1-7**) Edwards, S.F. and Gulyaev, Y.V.; Proc. Roy. Soc. (London), **1A279**, (1964), 229.
(**R1-8**) Berezin, F.A.; Theor. Math. Phys. **6**, (1971), 194.

The *mid-point rule* is derived as the mathematical consequence of the Weyl correspondence in the following article.

(**R1-9**) Mizrahi, M.M.; J. Math. Phys. **16**, (1975), 2201.

The application of the Weyl correspondence to Non-Abelian gauge field theory is discussed in the following article.

(**R1-10**) Christ, N.H., and Lee, T.D.; Phys. Rev. **D22**, (1980), 939.

Besides the above articles, we cite the following articles on the *operator-ordering problems* and the *effective potentials*.

(**R1-11**) Prokhorov, L.V.; Sov. J. Part. Nucl. **13**, (1982), 456; Sov. J. Nucl. Phys. **35**, (1982), 285.
(**R1-12**) Kappoor, A.K.; Phys. Rev. **D29**, (1984), 2339; **D30**, (1984), 1750.
(**R1-13**) Grosche, C. and Steiner, F.; Z. Phys. **C36**, (1987), 699.
(**R1-14**) Gadella, M.; Fort. Phys. **43**, (1995), 229.

A variety of path integrals for the potential problem in quantum mechanics which can be evaluated analytically in the curvilinear coordinate system and variety of the operator ordering prescriptions are discussed in the following book.

(**T-27**) Grosche, C. and Steiner, F.; ″Handbook of Feynman Path Integrals″, Springer, 1998, Heidelberg.

In the book cited above, the path integrals which can be evaluated in closed form are classified as *Gaussian Path Integral*, *Besselian Path Integral*, and *Legendrian Path Integral*. Also discussed in the book include the *non-relativistic perturbation theory* and the *semiclassical theory*.

# 2. Path Integral Representation of Quantum Field Theory

In this chapter, we carry out the translation of the results of Chap. 1 on the path integral representation of quantum mechanics to the path integral representation of quantum field theory. The coordinate operator $\hat{q}_{\mathrm{H}}(t)$ of the quantum mechanical system gets demoted to the $c$-number indices $(x, y, z)$ at the same level as the time index $t$. Instead of $\hat{q}_{\mathrm{H}}(t)$, we have the second-quantized field operator $\hat{\psi}(t, x, y, z)$ in quantum field theory. The necessity of the second-quantization of the field variable $\psi(t, x, y, z)$ originates from the various paradoxes encountered in relativistic quantum mechanics.

In Sect. 2.1, we discuss the path integral quantization of field theory. We discuss the review of classical field theory (Sect. 2.1.1), the phase space path integral quantization (Sect. 2.1.2), and the configuration space path integral quantization (Sect. 2.1.3). These are a translation of the results of Sect. 1.2 to the case of quantum field theory. As in Chap. 1, in the Feynman path integral formula as applied to quantum field theory, we shall use the effective Lagrangian density, instead of the original Lagrangian density, which takes the $\psi$-dependent determinant factor and the effective potential (or the effective interaction) into consideration.

In Sect. 2.2, we discuss covariant perturbation theory, using the path integral representation of the generating functional of the Green's functions (the vacuum expectation value of the time-ordered product of the field operators) derived in Sect. 2.1.3. We discuss the generating functional of the Green's functions of the free field (Sect. 2.2.1), the generating functional of the Green's functions of the interacting field (Sect. 2.2.2), and the Feynman–Dyson expansion and Wick's theorem (Sect. 2.2.3).

The procedure employed in Sect. 2.1 is the translation of the path integral representation of quantum mechanics to the path integral representation of quantum field theory. In Sect. 2.3, we discuss Symanzik construction of the path integral formula for the generating functional of the Green's functions directly from the canonical formalism of quantum field theory, and show that there are no errors in the translation. We derive the equation of motion satisfied by the generating functional of the Green's functions from the definition of the generating functional of the Green's functions, the Euler–Lagrange equation of motion satisfied by the field operators in the Heisenberg picture, and the canonical equal-time (anti-) commutator (Sect. 2.3.1). The equation

M. Masujima: Path Integral Quantization and Stochastic Quantization,
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of motion satisfied by the generating functional is the functional differential equation. We can solve this functional differential equation immediately with the use of the functional Fourier transform (Sect. 2.3.2). Ultimately, we reduce the problem to that of obtaining the Green's functions under the influence of the external field, with the example taken from the neutral ps-ps coupling (Sect. 2.3.3). We find that the result of Sect. 2.3 agrees with that of Sect. 2.1.

In Sect. 2.4, we derive the Schwinger–Dyson equation satisfied by the "full" Green's functions. We first derive part of the infinite system of equations of motion satisfied by the "full" Green's functions from the equation of motion of the generating functional of the "full" Green's functions and the definition of the "full" Green's functions (Sect. 2.4.1). Next, following Dyson, with the introduction of the proper self-energy parts and the vertex operator in the presence of the external hook, we shall derive Schwinger–Dyson equation satisfied by the "full" Green's functions, in exact and closed form (Sect. 2.4.2). By the method of Sect. 2.2, we can obtain the "full" Green's functions perturbatively, whereas by the method of Sect. 2.4, we can obtain the "full" Green's functions, proper self-energy parts and vertex operator all together iteratively from the Schwinger–Dyson equation. Thus, we have two methods of covariant perturbation theory, one based on the path integral representation, and the other based on the equation of motion of the generating functional of the "full" Green's functions.

In Sect. 2.5, we demonstrate the equivalence of path integral quantization and canonical quantization, starting from the former. We adopt the Feynman path integral formula for the transformation functions,

$$\langle q_{t_b}, t_b | q_{t_a}, t_a \rangle \text{ and } \langle \phi'', \sigma'' | \phi', \sigma' \rangle,$$

as Feynman's action principle (Sect. 2.5.1). We derive the definition of the field operator, the Euler–Lagrange equation of motion and the definition of the time-ordered product (Sect. 2.5.2), and the definition of the canonically conjugate momentum and the canonical equal-time (anti-)commutator (Sect. 2.5.3), from Feynman's action principle. In view of the conclusions of Sects. 2.1 and 2.3, we have demonstrated the equivalence of path integral quantization and canonical quantization for a nonsingular Lagrangian system. In this section, we discuss quantum mechanics and quantum field theory in parallel.

## 2.1 Path Integral Quantization of Field Theory

In this section, we discuss the path integral quantization of relativistic field theory based on the conclusions of Chap. 1. The major purpose of this section is to reach covariant perturbation theory (to be discussed in the next section) as quickly as possible, and hence our major task is the translation of a mechanical system with finite degrees of freedom to a multicomponent field system with infinite degrees of freedom. We translate as follows:

$$r \to (\boldsymbol{x}, n),$$

$$q_r(t) \to \psi_{\boldsymbol{x},n}(t) \equiv \psi_n(t, \boldsymbol{x}),$$

$$\sum_{r=1}^{f} \to \int \mathrm{d}^3\boldsymbol{x} \sum_{n=1}^{f} .$$

We briefly review classical field theory in order to establish relativistic notation (Sect. 2.1.1). We carry out the phase space path integral quantization of field theory (Sect. 2.1.2) and the configuration space path integral quantization of field theory (Sect. 2.1.3), following the translation stated above. We will find out that, instead of the original Lagrangian density

$$\mathcal{L}(\psi(x), \partial_\mu \psi(x)),$$

the effective Lagrangian density

$$\mathcal{L}_{\text{eff}}(\psi(x), \partial_\mu \psi(x)),$$

which takes the $\psi$-dependent determinant factor and the effective potential into consideration, should be used in the Feynman path integral formula.

In the above discussion, we use the symbol $\psi(x)$ as if it were the Bose field. If it were the Fermi field, we refer the reader to Appendices 2 and 3 for fermion number integration.

### 2.1.1 Review of Classical Field Theory

Keeping in mind relativistic field theory, we shall establish the relativistic notation first. We employ a natural unit system in which we have

$$\hbar = c = 1 . \tag{2.1.1}$$

We define the Minkowski space-time metric tensor $\eta_{\mu\nu}$ by

$$\eta_{\mu\nu} = \mathrm{diag}(1; -1, -1, -1) = \eta^{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, 3 . \tag{2.1.2}$$

We define the contravariant and covariant components of the space-time coordinates $x$ by

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (t; \boldsymbol{x}) , \tag{2.1.3a}$$

$$x_\mu = \eta_{\mu\nu} x^\nu = (t; -\boldsymbol{x}) . \tag{2.1.3b}$$

We define the differential operators $\partial_\mu$ and $\partial^\mu$ by

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}; \boldsymbol{\nabla}\right), \tag{2.1.4a}$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \eta^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}; -\boldsymbol{\nabla}\right). \tag{2.1.4b}$$

We define the four-scalar product by

$$x \cdot y \equiv x^\mu y_\mu = \eta_{\mu\nu} x^\mu y^\nu = x^0 \cdot y^0 - \boldsymbol{x} \cdot \boldsymbol{y}\,. \tag{2.1.5}$$

From this section onward, we adopt the convention that the Greek indices $\mu$, $\nu$, ... run over 0, 1, 2 and 3, the Latin indices $i$, $j$, ... run over 1,2 and 3 and repeated indices are summed over.

When we have the Lagrangian density $\mathcal{L}(\psi(x), \partial_\mu\psi(x))$, classical field theory is determined by the action functional $I[\psi]$ defined by

$$I[\psi] \equiv \int_\Omega \mathrm{d}^4x \mathcal{L}(\psi(x), \partial_\mu\psi(x))\,. \tag{2.1.6}$$

In response to variation of the field quantity,

$$\psi_n(x) \to \psi_n(x) + \delta\psi_n(x),$$

we have the variation of the action functional,

$$\begin{aligned}\delta I[\psi] &= \int_\Omega \mathrm{d}^4x \left\{\frac{\partial\mathcal{L}}{\partial\psi_n(x)}\delta\psi_n(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_n(x))}\delta(\partial_\mu\psi(x))\right\} \\ &= \int_\Omega \mathrm{d}^4x \left\{\frac{\partial\mathcal{L}}{\partial\psi_n(x))} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_n(x))}\right)\right\}\delta\psi_n(x) \\ &\quad + \int_{\partial\Omega} \mathrm{d}\sigma_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_n(x))}\delta\psi_n(x)\,.\end{aligned} \tag{2.1.7}$$

According to the principle of least action, we demand that

$$\frac{\delta I[\psi]}{\delta\psi_n(x)} = 0 \quad \text{in} \quad \Omega; \qquad \delta\psi_n(x) = 0 \quad \text{on} \quad \partial\Omega\,. \tag{2.1.8}$$

From (2.1.8), we obtain the Euler–Lagrange equation of motion,

$$\frac{\partial\mathcal{L}}{\partial\psi_n(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_n(x))}\right) = 0, \qquad n = 1, \ldots, f\,. \tag{2.1.9}$$

For the transition to the canonical formalism, we introduce the unit time-like vector

$$n_\mu = (1; 0, 0, 0),$$

and define the momentum $\pi_n(x)$ canonically conjugate to $\psi_n(x)$ by

$$\pi_n(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi_n(x))} \equiv n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi_n(x))}, \qquad n_\mu n^\mu = 1\,. \tag{2.1.10}$$

Solving (2.1.10) for $n^\mu\partial_\mu\psi_n(x) = \partial_0\psi_n(x)$ as a function of $\psi_n(x)$, $\boldsymbol{\nabla}\psi_n(x)$ and $\pi_n(x)$, we define the Hamiltonian density by

$$\mathcal{H}(\psi(x), \boldsymbol{\nabla}\psi(x), \pi(x)) \equiv \sum_{n=1}^{f} \pi_n(x)\partial_0\psi_n(x) - \mathcal{L}(\psi(x), \partial_\mu\psi(x))\,. \tag{2.1.11}$$

We define the Hamiltonian by

$$H[\psi(t), \boldsymbol{\nabla}\psi(t), \pi(t)] \equiv \int \mathrm{d}^3x\mathcal{H}(\psi(x), \boldsymbol{\nabla}\psi(x), \pi(x))\,. \tag{2.1.12}$$

We have the canonical equation of motion in classical field theory as

$$\partial_0\psi_n(x) = \frac{\delta H[\psi, \boldsymbol{\nabla}\psi, \pi]}{\delta\pi_n(x)}\,, \tag{2.1.13a}$$

$$\partial_0\pi_n(x) = -\frac{\delta H[\psi, \boldsymbol{\nabla}\psi, \pi]}{\delta\psi_n(x)}\,, \tag{2.1.13b}$$

where in (2.1.13a) and (2.1.13b), we have

$$n = 1, \ldots, f.$$

We note that $\pi_n(x)$ and $\mathcal{H}(\psi(x), \boldsymbol{\nabla}\psi(x), \pi(x))$ are normal dependent.

### 2.1.2 Phase Space Path Integral Quantization of Field Theory

We begin the discussion of the translation (hereafter designated by $\rightarrow$) from quantum mechanics with $f$ degrees of freedom to $f$-component quantum field theory with infinite degrees of freedom. We perform the above translation upon the canonical variables $\{\hat{q}_r(t)\}_{r=1}^{f}$, and $\{\hat{p}_r(t)\}_{r=1}^{f}$, their equal-time canonical commutators, their respective eigenkets $|q,t\rangle$ and $|p,t\rangle$, and the $q$-representation of $\hat{p}_r(t)$:

$$r \rightarrow (\boldsymbol{x}, n), \qquad n = 1, \ldots, f,$$

$$\hat{q}_r(t) \rightarrow \hat{\psi}_n(t, \boldsymbol{x}) = \hat{\psi}_{\boldsymbol{x},n}(t)\,, \tag{2.1.14a}$$

$$\hat{p}_r(t) \rightarrow \hat{\pi}_n(t, \boldsymbol{x}) = \hat{\pi}_{\boldsymbol{x},n}(t)\,, \tag{2.1.14b}$$

$$[\hat{q}_r(t), \hat{q}_s(t)] = [\hat{p}_r(t), \hat{p}_s(t)] = 0$$

$$\to [\hat{\psi}_n(t, \boldsymbol{x}), \hat{\psi}_m(t, \boldsymbol{y})] = [\hat{\pi}_n(t, \boldsymbol{x}), \hat{\pi}_m(t, \boldsymbol{y})] = 0, \tag{2.1.15a}$$

$$[\hat{q}_r(t), \hat{p}_s(t)] = \mathrm{i}\hbar\delta_{r,s}$$

$$\to [\hat{\psi}_n(t, \boldsymbol{x}), \hat{\pi}_m(t, \boldsymbol{y})] = \mathrm{i}\delta_{n,m}\delta^3(\boldsymbol{x} - \boldsymbol{y}), \tag{2.1.15b}$$

$$|q, t\rangle \to |\psi, t\rangle \quad \text{with} \quad \hat{\psi}_n(t, \boldsymbol{x})|\psi, t\rangle = \psi_n(t, \boldsymbol{x})|\psi, t\rangle\,, \tag{2.1.16a}$$

$$|p, t\rangle \to |\pi, t\rangle \quad \text{with} \quad \hat{\pi}_n(t, \boldsymbol{x})|\pi, t\rangle = \pi_n(t, \boldsymbol{x})|\pi, t\rangle\,, \tag{2.1.16b}$$

and

$$\langle q, t|\hat{p}_r(t)|\Phi\rangle = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q_r}\langle q, t|\Phi\rangle$$

$$\to \langle \psi, t|\hat{\pi}_n(t, \boldsymbol{x})|\Phi\rangle = \frac{1}{\mathrm{i}}\frac{\delta}{\delta\psi_n(t, \boldsymbol{x})}\langle \psi, t|\Phi\rangle\,. \tag{2.1.17}$$

With these translations, (2.1.14a) through (2.1.17), we have the following equations corresponding to (1.2.17), (1.2.20), (1.2.34) and (1.2.39).

**Transformation Function**:

$$\langle \psi', t'|\psi, t\rangle = \int_{\psi(t)=\psi}^{\psi(t')=\psi'} \mathcal{D}[\psi] \int \mathcal{D}[\pi] \times \exp\left[\mathrm{i}\int_t^{t'} \mathrm{d}t \left\{\int \mathrm{d}^3\boldsymbol{x} \sum_{n=1}^{f} \pi_n(t, \boldsymbol{x})\partial_0\psi_n(t, \boldsymbol{x}) - H[\psi(t), \boldsymbol{\nabla}\psi(t), \pi(t)]\right\}\right]. \tag{2.1.18}$$

**Matrix Element of the Time-Ordered Product**:

$$\langle \psi', t'|\mathrm{T}(\hat{\psi}_{n_1}(x_1)\cdots\hat{\psi}_{n_k}(x_k))|\psi, t\rangle = \int_{\psi(t)=\psi}^{\psi(t')=\psi'} \mathcal{D}[\psi] \int \mathcal{D}[\pi] \times \psi_{n_1}(x_1)\cdots\psi_{n_k}(x_k) \times \exp\left[\mathrm{i}\int_t^{t'} \mathrm{d}t \left\{\int \mathrm{d}^3\boldsymbol{x} \sum_{n=1}^{f} \pi_n(t, \boldsymbol{x})\partial_0\psi_n(t, \boldsymbol{x}) - H[\psi(t), \boldsymbol{\nabla}\psi(t), \pi(t)]\right\}\right]. \tag{2.1.19}$$

**Wave Functions of the Vacuum**:

$$\langle\psi_{\pm\infty},\pm\infty|0;^{\text{out}}_{\text{in}}\rangle$$
$$=\exp\left[-\frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\int \mathrm{d}^3\boldsymbol{y}\sum_{n=1}^{f}\psi_n(\pm\infty,\boldsymbol{x})E(\boldsymbol{x}-\boldsymbol{y})\psi_n(\pm\infty,\boldsymbol{y})\right], \quad (2.1.20a)$$

$$E(\boldsymbol{x}-\boldsymbol{y})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\exp\left[\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})\right]E_{\boldsymbol{p}}, \quad E_{\boldsymbol{p}}=\sqrt{\boldsymbol{p}^2+m^2}\,. \quad (2.1.20b)$$

**Generating Functional of the Green's Functions**:

$$\begin{aligned}
Z[J]&=\exp[\mathrm{i}W[J]]\\
&=\left\langle 0;\text{out}\left|\mathrm{T}\left(\exp\left[\mathrm{i}\int \mathrm{d}^4x\sum_{n=1}^{f}J_n(x)\hat{\psi}_n(x)\right]\right)\right|0;\text{in}\right\rangle\\
&=\int \mathrm{d}\psi_{+\infty}\int \mathrm{d}\psi_{-\infty}\langle 0;\text{out}|\psi_{+\infty},+\infty\rangle\\
&\quad\times\left\langle\psi_{+\infty},+\infty\left|\mathrm{T}\left(\exp\left[\mathrm{i}\int \mathrm{d}^4x\sum_{n=1}^{f}J_n(x)\hat{\psi}_n(x)\right]\right)\right|\psi_{-\infty},-\infty\right\rangle\\
&\quad\times\langle\psi_{-\infty},-\infty|0;\text{in}\rangle\\
&=\int\mathcal{D}[\psi]\int\mathcal{D}[\pi]\exp\left[\mathrm{i}\int_{-\infty}^{+\infty}\mathrm{d}t\left(\left(\int \mathrm{d}^3\boldsymbol{x}\sum_{n=1}^{f}\pi_n(t,\boldsymbol{x})\partial_0\psi_n(t,\boldsymbol{x})\right.\right.\right.\\
&\quad\left.-H[\psi(t),\boldsymbol{\nabla}\psi(t),\pi(t)]+\int \mathrm{d}^3\boldsymbol{x}J_n(t,\boldsymbol{x})\psi_n(t,\boldsymbol{x})\right)\\
&\quad\left.\left.+\text{ ``i}\varepsilon\text{-piece''}\right)\right]. \qquad (2.1.21)
\end{aligned}$$

Here, we write down the "i$\varepsilon$-piece" explicitly,

$$\begin{aligned}
\text{``i}\varepsilon\text{-piece''}&=\mathrm{i}\frac{\varepsilon}{2}\int \mathrm{d}^3\boldsymbol{x}\int \mathrm{d}^3\boldsymbol{y}E(\boldsymbol{x}-\boldsymbol{y})\\
&\quad\times\sum_{n=1}^{f}\psi_n(t,\boldsymbol{x})\psi_n(t,\boldsymbol{y})\exp[-\varepsilon|t|],\ \varepsilon\to 0^+\,, \qquad (2.1.22)
\end{aligned}$$

which is $t$-dependent. We note that the path integral with respect to $\psi_n(t,\boldsymbol{x})$

$$\int\mathcal{D}[\psi]$$

in (2.1.21) is no longer constrained. $Z[J]$ ($W[J]$) is the generating functional of (the connected parts of) the vacuum expectation values of the time-ordered products of the field operator $\hat{\psi}_{n_k}(x_k)$. We have the formula

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_{n_k}(x_k)}\cdots\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_{n_1}(x_1)}Z[J]|_{J=0}$$

$$=\langle 0;\mathrm{out}|\mathrm{T}(\hat{\psi}_{n_1}(x_1)\cdots\hat{\psi}_{n_k}(x_k))|0;\mathrm{in}\rangle\,, \tag{2.1.23Z}$$

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_{n_k}(x_k)}\cdots\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_{n_1}(x_1)}W[J]|_{J=0}$$

$$=\frac{1}{\mathrm{i}}\langle 0;\mathrm{out}|\mathrm{T}(\hat{\psi}_{n_1}(x_1)\cdots\hat{\psi}_{n_k}(x_k))|0;\mathrm{in}\rangle_{\mathrm{C}}\,, \tag{2.1.23W}$$

where (2.1.23W) defines the connected parts.

### 2.1.3 Configuration Space Path Integral Quantization of Field Theory

In order to discuss the configuration space path integral representation of quantum field theory, we assume that the Hamiltonian $H[\psi(t),\boldsymbol{\nabla}\psi(t),\pi(t)]$ defined by (2.1.12) is given by a quadratic form in $\pi_n(x)$, namely

$$\begin{aligned}H[\psi(t),\boldsymbol{\nabla}\psi(t),\pi(t)]\equiv{}&\frac{1}{2}\int\mathrm{d}^3\boldsymbol{x}\int\mathrm{d}^3\boldsymbol{y}\pi_n(t,\boldsymbol{x})D_{n,m}(\boldsymbol{x},\boldsymbol{y};\psi)\pi_m(t,\boldsymbol{y})\\&+\text{a term linear in }\pi\\&+\text{a term independent of }\pi\,.\end{aligned} \tag{2.1.24}$$

Then, we can perform the momentum integration

$$\int\mathcal{D}[\pi]$$

in (2.1.18), (2.1.19) and (2.1.21) as a quasi-Gaussian integral, obtaining the determinant factor

$$(\mathrm{Det}D_{n,m}(\boldsymbol{x},\boldsymbol{y};\psi))^{-1/2}$$

in the integrand of

$$\int\mathcal{D}[\psi].$$

We have the stationary condition of the exponent of (2.1.18), (2.1.19) and (2.1.21) with respect to $\pi_n(x)$ as

$$\frac{\delta}{\delta\pi_n(t,\boldsymbol{x})}\left\{\int \mathrm{d}^3x' \sum_{m=1}^{f} \pi_m(t,\boldsymbol{x}')\partial_0\psi_m(t,\boldsymbol{x}') - H[\psi(t),\boldsymbol{\nabla}\psi(t),\pi(t)]\right\} = 0\,, \tag{2.1.25}$$

i.e., we have one of a pair of canonical equations of motion,

$$\partial_0\psi_n(t,\boldsymbol{x}) = \frac{\delta H[\psi,\boldsymbol{\nabla}\psi,\pi]}{\delta\pi_n(t,\boldsymbol{x})}\,. \tag{2.1.26}$$

Solving (2.1.26) for $\pi_n(t,\boldsymbol{x})$, we obtain the canonical definition of $\pi_n(t,\boldsymbol{x})$. Hence, we obtain the Lagrangian $L[\psi(t),\partial_0\psi(t)]$ as the stationary value of the exponent

$$\begin{aligned}
&\int \mathrm{d}^3\boldsymbol{x} \sum_{m=1}^{f} \pi_m(t,\boldsymbol{x})\partial_0\psi(t,\boldsymbol{x}) \\
&\qquad -H[\psi(t),\boldsymbol{\nabla}\psi(t),\pi(t)]|_{\partial_0\psi_n(t,\boldsymbol{x})=\delta H[\psi,\boldsymbol{\nabla}\psi,\pi]/\delta\pi_n(t,\boldsymbol{x})} \\
&\quad \equiv L[\psi(t),\partial_0\psi(t)] \equiv \int \mathrm{d}^3\boldsymbol{x}\mathcal{L}(\psi(x),\partial_\mu\psi(x))\,.
\end{aligned} \tag{2.1.27}$$

From these considerations, we obtain the configuration space path integral formula for the transformation function, the matrix element of the time-ordered product and the generating functional of (the connected parts of) the vacuum expectation value of the time-ordered product:

**Transformation Function**:

$$\begin{aligned}
\langle\psi',t'|\psi,t\rangle = &\int_{\psi(t)=\psi}^{\psi(t')=\psi'} \mathcal{D}[\psi](\mathrm{Det}D(\boldsymbol{x},\boldsymbol{y};\psi))^{-1/2} \\
&\times \exp\left[\mathrm{i}\int_t^{t'} \mathrm{d}tL[\psi(t),\partial_0\psi(t)]\right].
\end{aligned} \tag{2.1.28}$$

**Matrix Element of the Time-Ordered Product**:

$$\begin{aligned}
&\langle\psi',t'|\mathrm{T}(\hat{\psi}_{n_1}(x_1)\cdots\hat{\psi}_{n_k}(x_k))|\psi,t\rangle \\
&\quad = \int_{\psi(t)=\psi}^{\psi(t')=\psi'} \mathcal{D}[\psi](\mathrm{Det}D(\boldsymbol{x},\boldsymbol{y};\psi))^{-1/2}\psi_{n_1}(x_1)\cdots\psi_{n_k}(x_k) \\
&\qquad \times \exp\left[\mathrm{i}\int_t^{t'} \mathrm{d}tL[\psi(t),\partial_0\psi(t)]\right].
\end{aligned} \tag{2.1.29}$$

**Generating Functional of the Green's Functions**:

$$\begin{aligned}
Z[J] &= \exp[\mathrm{i}W[J]] \\
&= \left\langle 0;\mathrm{out}\left|\mathrm{T}\left(\exp\left[\int \mathrm{d}^4x \sum_{n=1}^{f} J_n(x)\hat{\psi}_n(x)\right]\right)\right|0;\mathrm{in}\right\rangle \\
&= \int \mathcal{D}[\psi](\mathrm{Det}D(\boldsymbol{x},\boldsymbol{y};\psi))^{-1/2} \\
&\quad\times \exp\left[\mathrm{i}\int \mathrm{d}^4x\left\{\mathcal{L}(\psi(x),\partial_\mu\psi(x))\right.\right. \\
&\qquad\left.\left. + \sum_{n=1}^{f} J_n(x)\psi_n(x) + \text{“i}\varepsilon\text{-piece”}\right\}\right]. \qquad (2.1.30)
\end{aligned}$$

Equations (2.1.28–2.1.30) are the field theory version of the results of Sect. 1.2. We note that only when the kernel of the quadratic part of (2.1.24),

$$\{D_{n,m}(\boldsymbol{x},\boldsymbol{y};\psi)\}_{n,m=1}^{f},$$

is a $\psi_n(x)$-independent constant matrix, is a naive extension of the Feynman path integral formula in quantum mechanics to quantum field theory correct.

When the kernel $\{D_{n,m}(\boldsymbol{x},\boldsymbol{y};\psi)\}_{n,m=1}^{f}$ is a $\psi_n(x)$-dependent matrix, and is diagonal with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$D_{n,m}(\boldsymbol{x},\boldsymbol{y};\psi) = \delta^3(\boldsymbol{x}-\boldsymbol{y})D_{n,m}(\psi(t,\boldsymbol{x})), \qquad n,m=1,\ldots,f\,, \quad (2.1.31)$$

we obtain

$$\mathrm{Det}D_{n,m}(\boldsymbol{x},\boldsymbol{y};\psi) = \exp\left[\delta^4(0)\int \mathrm{d}^4x \mathrm{tr}\ln D(\psi(x))\right], \qquad (2.1.32\mathrm{a})$$

by an argument similar to that leading to (1.124) from (1.122). We shall derive (2.1.32a) in some detail here. With $D_{n,m}(\boldsymbol{x},\boldsymbol{y};\psi)$ diagonal with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$, we have $\ln D(\boldsymbol{x},\boldsymbol{y};\psi)$ diagonal with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$\ln D_{n,,,m}(\boldsymbol{x},\boldsymbol{y};\psi) = \delta^3(\boldsymbol{x}-\boldsymbol{y})\ln D_{n,m}(\psi(t,\boldsymbol{x}))\,. \qquad (2.1.33)$$

Next, we apply (1.2.51) at each time slice $t=t_k, k=1,\ldots,l$, obtaining

$$\begin{aligned}
\mathrm{Det}D(\boldsymbol{x},\boldsymbol{y};\psi) &= \lim_{\substack{l\to\infty \\ \delta t\to 0}} \prod_{k=1}^{l} \mathrm{Det}D(\boldsymbol{x}_k,\boldsymbol{y}_k;\psi(t_k,\boldsymbol{x}_k)) \\
&= \lim_{\substack{l\to\infty \\ \delta t\to 0}} \exp\left[\sum_{k=1}^{l} \mathrm{Tr}\ln D(\boldsymbol{x}_k,\boldsymbol{y}_k;\psi(t_k,\boldsymbol{x}_k))\right]
\end{aligned}$$

$$= \lim_{\substack{l\to\infty \\ \delta t\to 0}} \exp\left[\sum_{k=1}^{l} \delta^3(0) \int \mathrm{d}^3\boldsymbol{x} \mathrm{tr} \ln D(\psi(t_k, \boldsymbol{x}))\right]$$

$$= \lim_{\substack{l\to\infty \\ \delta t\to 0}} \exp\left[\delta^3(0) \frac{1}{\delta t} \sum_{k=1}^{l} \delta t \int \mathrm{d}^3\boldsymbol{x} \mathrm{tr} \ln D(\psi(t_k, \boldsymbol{x}))\right]$$

$$= \exp\left[\delta^4(0) \int \mathrm{d}^4 x \mathrm{tr} \ln D(\psi(x))\right], \tag{2.1.32b}$$

which is (2.1.32a). Hence, in each integrand of (2.1.28–2.1.30), we have

$$(\mathrm{Det} D(\boldsymbol{x}, \boldsymbol{y}; \psi))^{-1/2} \exp\left[\mathrm{i} \int \mathrm{d}^4 x \mathcal{L}(\psi(x), \partial_\mu \psi(x))\right]$$

$$= \exp\left[\mathrm{i} \int \mathrm{d}^4 x \left\{\mathcal{L}(\psi(x), \partial_\mu \psi(x)) - \mathrm{i}\frac{1}{2}\delta^4(0) \mathrm{tr} \ln D^{-1}(\psi(x))\right\}\right].$$

Thus, when the kernel of the quadratic part is a $\psi$-dependent matrix, and is diagonal with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$, we shall use the *effective Lagrangian density*

$$\mathcal{L}_{\mathrm{eff}}(\psi(x), \partial_\mu \psi(x)) = \mathcal{L}(\psi(x), \partial_\mu \psi(x)) - \mathrm{i}\frac{1}{2}\delta^4(0) \mathrm{tr} \ln D^{-1}(\psi(x)) \tag{2.1.34}$$

in the Feynman path integral formula, instead of the *original Lagrangian density*

$$\mathcal{L}(\psi(x), \partial_\mu \psi(x)).$$

Furthermore, according to the conclusion of Sect. 1.3, we have to consider the effective potential $V_{\mathrm{c}}(\psi(x))$ depending on whether $\psi(x)$ is curvilinear or Cartesian.

For the remainder of this subsection, we consider *chiral dynamics* as an example and calculate its effective Lagrangian density $\mathcal{L}_{\mathrm{eff}}(\psi(x), \partial_\mu \psi(x))$. We have the Lagrangian density for chiral dynamics as

$$\mathcal{L}(\psi(x), \partial_\mu \psi(x)) = \frac{1}{2}\partial_\mu \psi_a(x) G_{a,b}(\psi(x)) \partial^\mu \psi_b(x). \tag{2.1.35}$$

We have the canonical conjugate momentum $\pi_a(x)$ and the Hamiltonian density $\mathcal{H}(\psi_a(x), \boldsymbol{\nabla}\psi_a(x), \pi_a(x))$ as

$$\pi_a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a(x))} = G_{a,b}(\psi(x)) \partial^0 \psi_b(x), \tag{2.1.36}$$

and

$$\begin{aligned}\mathcal{H}(\psi_a(x),\boldsymbol{\nabla}\psi_a(x),\pi_a(x)) &= \pi_a(x)\partial_0\psi_a(x) - \mathcal{L}(\psi_a(x),\partial_\mu\psi_a(x)) \\ &= \frac{1}{2}\pi_a(x)G^{-1}_{a,b}(\psi(x))\pi_b(x) \\ &\quad -\frac{1}{2}\partial_k\psi_a(x)G_{a,b}(\psi(x))\partial^k\psi_b(x)\,. \end{aligned}\tag{2.1.37}$$

Thus, we have

$$\begin{aligned} &H[\psi,\boldsymbol{\nabla}\psi,\pi]\Big|^{\text{quadratic part in}}_{\text{momentum }\pi} \\ &\quad= \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\pi_a(x)G^{-1}_{a,b}(\psi(x))\pi_b(x) \\ &\quad= \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\int \mathrm{d}^3\boldsymbol{y}\pi_a(t,\boldsymbol{x})\delta^3(\boldsymbol{x}-\boldsymbol{y})G^{-1}_{a,b}(\psi(t,\boldsymbol{x}))\pi_b(t,\boldsymbol{y})\,. \end{aligned}\tag{2.1.38}$$

We find that the kernel $D_{n,m}(\boldsymbol{x},\boldsymbol{y};\psi)$ of (2.1.24) is given by

$$D_{a,b}(\boldsymbol{x},\boldsymbol{y};\psi(t,\boldsymbol{x})) = \delta^3(\boldsymbol{x}-\boldsymbol{y})G^{-1}_{a,b}(\psi(t,\boldsymbol{x}))\,,\tag{2.1.39a}$$

$$D_{a,b}(\psi(t,\boldsymbol{x})) = G^{-1}_{a,b}(\psi(t,\boldsymbol{x}))\,,\tag{2.1.39b}$$

$$D^{-1}_{a,b}(\psi(t,\boldsymbol{x})) = G_{a,b}(\psi(t,\boldsymbol{x}))\,.\tag{2.1.39c}$$

We know that the exponent of the integrand of the phase space path integral formula

$$\pi_a(x)\partial_0\psi_a(x) - \mathcal{H}(\psi_a(x),\boldsymbol{\nabla}\psi_a(x),\pi_a(x))\,,\tag{2.1.40}$$

is quadratic in $\pi_a(x)$ so that we can perform the momentum integration as a quasi-Gaussian integral. As the stationary condition of (2.1.40) with respect to $\pi_a(x)$, we have

$$\partial_0\psi_a(x) = \frac{\partial\mathcal{H}}{\partial\pi_a(x)}\,,\tag{2.1.41}$$

which is one of a pair of canonical equations of motion. Hence, we find that the stationary value of (2.1.40) is the original Lagrangian density $\mathcal{L}(\psi(x),\partial_\mu\psi(x))$. We can apply the argument leading to (2.1.34) to this problem. We find that the effective Lagrangian density $\mathcal{L}_{\mathrm{eff}}(\psi(x),\partial_\mu\psi(x))$ of chiral dynamics is given by

$$\mathcal{L}_{\mathrm{eff}}(\psi(x),\partial_\mu\psi(x)) = \mathcal{L}(\psi(x),\partial_\mu\psi(x)) - \mathrm{i}\frac{1}{2}\delta^4(0)\mathrm{tr}\ln G(\psi(x))\,.\tag{2.1.42}$$

This conclusion agrees with the result of J.M. Charap, based on the phase space path integral approach; that of I.S. Gerstein, R. Jackiw, B.W. Lee and

S. Weinberg, based on the canonical formalism; that of K. Nishijima and T. Watanabe, based on the generating functionals of the T-ordered product and T*-ordered product; and that of J. Honerkamp and K. Meets, based on the general relativity of pion coordinates.

## 2.2 Covariant Perturbation Theory

In this section, we discuss covariant perturbation theory. We use the configuration space path integral representation of the generating functional $Z[J]$ of the Green's functions, (2.1.30), as the starting point. In standard operator field theory, we first introduce the interaction picture, and then, by consideration of the Tomonager–Schwinger equation and its integrability condition, we derive covariant perturbation theory after a lengthy discussion. We can derive covariant perturbation theory easily and quickly from the configuration space path integral representation of the generating functional $Z[J]$ of the Green's functions. We can also derive covariant perturbation theory of singular Lagrangian field theory with an infinite dimensional invariance group by an extension of the method of the present section. In this respect, we shall discuss the path integral quantization of the singular Lagrangian field theory, specifically, the non-Abelian gauge field theory in Chap. 3 in detail.

In deriving covariant perturbation theory, we first separate by hand the total Lagrangian density $\mathcal{L}_{\text{tot}}(\psi(x), \partial_\mu\psi(x))$ of the interacting system into the quadratic part, $\mathcal{L}_{\text{quad}}(\psi(x), \partial_\mu\psi(x))$, and the nonquadratic part, $\mathcal{L}_{\text{int}}(\psi(x))$, with respect to $\psi(x)$. As a result of this separation, we obtain the generating functional $Z[J]$ of the Green's functions of the interacting system as

$$Z[J] = \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\text{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right] Z_0[J],$$

where we have the generating functional $Z_0[J]$ of the Green's functions of the "free" field as

$$Z_0[J] = \int \mathcal{D}[\psi]\exp\left[\mathrm{i}\int \mathrm{d}^4x\left\{\mathcal{L}_{\text{quad}}(\psi(x), \partial_\mu\psi(x)) + \sum_{n=1}^{f}\psi_n(x)J_n(x) + \text{“i}\varepsilon\text{-piece”}\right\}\right].$$

We derive these expressions in Sects. 2.2.1 and 2.2.2. We will easily read off Feynman rules and the Feynman–Dyson expansion of the Green's functions of the interacting system from the interaction Lagrangian density $\mathcal{L}_{\text{int}}(\psi(x))$, to be discussed in Sect. 2.2.3.

### 2.2.1 Generating Functional of Green's Functions of the Free Field

We consider an interacting system which consists of the real scalar field $\phi(x)$ and the Dirac spinor field $\psi(x)$. As the total Lagrangian density, we consider

$$\begin{aligned}\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{quad}}(\phi(x), \partial_\mu\phi(x), \psi(x), \partial_\mu\psi(x), \bar{\psi}(x), \partial_\mu\bar{\psi}(x)) \\ +\mathcal{L}_{\text{int}}(\phi(x), \psi(x), \bar{\psi}(x)) \, , \end{aligned} \tag{2.2.1a}$$

where $\mathcal{L}_{\text{quad}}$ consists of the quadratic part of $\phi(x)$, $\psi(x)$ and $\bar{\psi}(x)$ including "i$\varepsilon$-piece" and $\mathcal{L}_{\text{int}}$ is the interaction Lagrangian density:

$$\begin{aligned}\mathcal{L}_{\text{quad}} = \mathcal{L}_{\text{quad}}^{\text{Real scalar}}(\phi(x), \partial_\mu\phi(x)) \\ +\mathcal{L}_{\text{quad}}^{\text{Dirac}}(\psi(x), \partial_\mu\psi(x), \bar{\psi}(x), \partial_\mu\bar{\psi}(x)) \, , \end{aligned} \tag{2.2.1b}$$

$$\begin{aligned}\mathcal{L}_{\text{quad}}^{\text{Real scalar}} &= \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \kappa^2\phi^2 + \mathrm{i}\varepsilon\phi^2) \\ &= \frac{1}{2}\phi(-\partial^2 - \kappa^2 + \mathrm{i}\varepsilon)\phi \equiv \frac{1}{2}\phi K\phi \, , \end{aligned} \tag{2.2.1c}$$

$$\begin{aligned}\mathcal{L}_{\text{quad}}^{\text{Dirac}} &= \frac{1}{4}[\bar{\psi}_\alpha(x), D_{\alpha\beta}(x)\psi_\beta(x)] + \frac{1}{4}\left[D_{\beta\alpha}^{\mathrm{T}}(-x)\bar{\psi}_\alpha(x), \psi_\beta(x)\right] \\ &\equiv \bar{\psi}_\alpha D_{\alpha\beta}\psi_\beta \, , \end{aligned} \tag{2.2.1d}$$

$$\begin{aligned}K(x-y) &= \delta^4(x-y)(-\partial_x^2 - \kappa^2 + \mathrm{i}\varepsilon), \\ K^{-1}(x-y) &= \frac{\delta^4(x-y)}{-\partial_x^2 - \kappa^2 + \mathrm{i}\varepsilon} \, , \end{aligned} \tag{2.2.1e}$$

$$\begin{aligned}D_{\alpha\beta}(x-y) &= \delta^4(x-y)(\mathrm{i}\gamma_\mu\partial_x^\mu - m + \mathrm{i}\varepsilon), \\ D_{\alpha\beta}^{-1}(x-y) &= \delta^4(x-y)\left(\frac{1}{\mathrm{i}\gamma_\mu\partial_x^\mu - m + \mathrm{i}\varepsilon}\right)_{\alpha\beta} \, , \end{aligned} \tag{2.2.1f}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0; \qquad \bar{\psi}(x) = \psi^\dagger(x)\gamma^0 \, , \tag{2.2.1g}$$

$$\mathcal{L}_{\text{int}}(\phi(x), \psi(x), \bar{\psi}(x)) \equiv \mathcal{L}_{\text{tot}} - \mathcal{L}_{\text{quad}} \, . \tag{2.2.1h}$$

According to (2.1.30), we have the generating functionals of the "free" Green's functions as

$$\begin{aligned} Z_0[J] &= \exp[\mathrm{i}W_0[J]] \\ &= \int \mathcal{D}[\phi] \exp\left[\mathrm{i}\int \mathrm{d}^4x \left\{\mathcal{L}_{\text{quad}}^{\text{Real scalar}}(\phi(x), \partial_\mu\phi(x)) + J(x)\phi(x)\right\}\right] \\ &= \int \mathcal{D}[\phi] \exp\left[\mathrm{i}\left(\frac{1}{2}\phi K\phi + J\phi\right)\right] \\ &= \exp\left[-\frac{\mathrm{i}}{2}JK^{-1}J\right], \end{aligned} \tag{2.2.2}$$

$$\begin{aligned} &Z_0[\bar{\eta}, \eta] \\ &\quad= \exp\left[\mathrm{i}W_0[\bar{\eta}, \eta]\right] \\ &\quad= \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \exp\left[\mathrm{i}\int \mathrm{d}^4x \left\{\mathcal{L}_{\text{quad}}^{\text{Dirac}}(\psi(x), \partial_\mu\psi(x), \bar{\psi}(x), \partial_\mu\bar{\psi}(x))\right.\right. \\ &\qquad\qquad \left.\left. +\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right\}\right] \\ &\quad= \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \exp\left[\mathrm{i}(\bar{\psi}_\alpha D_{\alpha\beta}\psi_\beta + \bar{\eta}_\beta\psi_\beta + \bar{\psi}_\alpha\eta_\alpha)\right] \\ &\quad= \exp\left[-\mathrm{i}\bar{\eta}_\beta D_{\beta\alpha}^{-1}\eta_\alpha\right]. \end{aligned} \tag{2.2.3}$$

We have the generating functionals, $W_0[J]$ and $W_0[\bar{\eta}, \eta]$, of the connected parts of the "free" Green's functions as

$$W_0[J] = -\frac{1}{2}JK^{-1}J = -\frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y J(x)K^{-1}(x-y)J(y)\,, \tag{2.2.4}$$

$$W_0[\bar{\eta}, \eta] = -\bar{\eta}_\beta D_{\beta\alpha}^{-1}\eta_\alpha = -\int \mathrm{d}^4x \int \mathrm{d}^4y \bar{\eta}_\beta(x) D_{\beta\alpha}^{-1}(x-y)\eta_\alpha(y)\,. \tag{2.2.5}$$

In writing down the last equalities of (2.2.2) and (2.2.3), we dropped the constants

$$(\mathrm{Det}K(x-y))^{-1/2}, \qquad \text{and} \qquad \mathrm{Det}D(x-y),$$

which are independent of $\phi(x)$, $\psi(x)$ and $\bar{\psi}(x)$. From (2.2.2) through (2.2.5), we have the "free" Green's functions as

$$
\begin{aligned}
&D_0^{\mathrm{F}}(x_1 - x_2) \\
&\quad = G_0(x_1 - x_2) = \frac{1}{\mathrm{i}} \langle 0, \mathrm{out} | \mathrm{T}(\hat{\phi}(x_1)\hat{\phi}(x_2)) | 0, \mathrm{in} \rangle_{\mathrm{C}}^{\mathrm{free}} \\
&\quad = \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x_1)} \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x_2)} W_0[J]|_{J=0} = K^{-1}(x_1 - x_2) \\
&\quad = \frac{\delta^4(x_1 - x_2)}{-\partial_{x_1}^2 - \kappa^2 + \mathrm{i}\varepsilon} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\exp\left[-\mathrm{i}p \cdot (x_1 - x_2)\right]}{p^2 - \kappa^2 + \mathrm{i}\varepsilon} ,
\end{aligned}
\tag{2.2.6}
$$

$$
\begin{aligned}
&S_{0,\alpha\beta}^{\mathrm{F}}(x_1 - x_2) = G_{0,\alpha\beta}(x_1 - x_2) \\
&\quad = \frac{1}{\mathrm{i}} \langle 0, \mathrm{out} | \mathrm{T}(\hat{\psi}_\alpha(x_1)(\hat{\psi}^\dagger(x_2)\gamma^0)_\beta) | 0, \mathrm{in} \rangle_{\mathrm{C}}^{\mathrm{free}} \\
&\quad = \frac{1}{\mathrm{i}} \frac{\delta}{\delta \bar{\eta}_\alpha(x_1)} \frac{1}{\mathrm{i}} \frac{\delta}{\delta \eta_\beta(x_2)} W_0[\bar{\eta}, \eta]|_{\eta=\bar{\eta}=0} = D_{\alpha\beta}^{-1}(x_1 - x_2) \\
&\quad = \delta^4(x_1 - x_2) \left( \frac{1}{\mathrm{i}\gamma_\mu \partial_{x_1}^\mu - m + \mathrm{i}\varepsilon} \right)_{\alpha\beta} \\
&\quad = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left( \frac{\exp[-\mathrm{i}p \cdot (x_1 - x_2)]}{\not{p} - m + \mathrm{i}\varepsilon} \right)_{\alpha\beta} .
\end{aligned}
\tag{2.2.7}
$$

### 2.2.2 Generating Functional of Full Green's Functions of an Interacting System

According to (2.1.30), we have the generating functional of the "full" Green's functions and its connected parts of the interacting system as

$$
\begin{aligned}
&Z[J, \bar{\eta}, \eta] \\
&\quad = \exp[\mathrm{i}W[J, \bar{\eta}, \eta]] \\
&\quad = \left\langle 0, \mathrm{out} \middle| \mathrm{T} \left( \exp \left[ \mathrm{i} \int \mathrm{d}^4 x \left\{ J(x)\hat{\phi}(x) + \bar{\eta}_\alpha(x)\hat{\psi}_\alpha(x) \right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. + (\hat{\psi}^\dagger(x)\gamma^0)_\beta \eta_\beta(x) \right\} \right] \right) \middle| 0, \mathrm{in} \right\rangle \\
&\quad = \int \mathcal{D}[\phi] \int \mathcal{D}[\psi] \int \mathcal{D}[\bar{\psi}] \exp \left[ \mathrm{i} \int \mathrm{d}^4 x \left\{ \mathcal{L}_{\mathrm{tot}}(x) + J(x)\phi(x) \right.\right. \\
&\qquad \left.\left. + \bar{\eta}_\alpha(x)\psi_\alpha(x) + \bar{\psi}_\beta(x)\eta_\beta(x) \right\} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}(x)}, \mathrm{i}\frac{\delta}{\delta\eta(x)}\right)\right] \\
&\quad\times\int\mathcal{D}[\phi]\exp\left[\mathrm{i}\int \mathrm{d}^4x\left\{\mathcal{L}_{\mathrm{quad}}^{\mathrm{Real\ scalar}}(\phi(x), \partial_\mu\phi(x)) + J(x)\phi(x)\right\}\right] \\
&\quad\times\int\mathcal{D}[\psi]\int\mathcal{D}[\bar{\psi}] \\
&\quad\times\exp\left[\mathrm{i}\int \mathrm{d}^4x\left\{\mathcal{L}_{\mathrm{quad}}^{\mathrm{Dirac}}(\psi(x), \partial_\mu\psi(x), \bar{\psi}(x), \partial_\mu\bar{\psi}(x))\right.\right. \\
&\quad\left.\left.+\bar{\eta}_\alpha(x)\psi_\alpha(x) + \bar{\psi}_\beta(x)\eta_\beta(x)\right\}\right] \\
&= \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}(x)}, \mathrm{i}\frac{\delta}{\delta\eta(x)}\right)\right] \\
&\quad\times Z_0[J]Z_0[\bar{\eta}, \eta]\,. \qquad (2.2.8)
\end{aligned}
$$

Generally, we have the connected part of the "full" Green's function (also called the $(l+m+n)$-point function) as,

$$
\begin{aligned}
&G^{\mathrm{C}}_{\alpha_1,\ldots,\alpha_l;1,\ldots,m;\beta_1,\ldots,\beta_n}(x_1,\ldots,x_l;y_1,\ldots,y_m;z_1,\ldots,z_n) \\
&\quad= \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y_1)}\cdots\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y_m)}\cdot\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_{\alpha_1}(x_1)}\cdots\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_{\alpha_l}(x_l)} \\
&\quad\quad\times\mathrm{i}\frac{\delta}{\delta\eta_{\beta_n}(z_n)}\cdots\mathrm{i}\frac{\delta}{\delta\eta_{\beta_1}(z_1)}\cdot W[J,\bar{\eta},\eta]|_{J=\bar{\eta}=\eta=0} \\
&\quad= \frac{1}{\mathrm{i}}\langle 0,\mathrm{out}|\mathrm{T}(\hat{\psi}_{\alpha_1}(x_1)\cdots\hat{\psi}_{\alpha_l}(x_l)(\hat{\psi}^\dagger(z_1)\gamma^0)_{\beta_1}\cdots(\hat{\psi}^\dagger(z_n)\gamma^0)_{\beta_n} \\
&\quad\quad\times\hat{\phi}(y_1)\cdots\hat{\phi}(y_m))|0,\mathrm{in}\rangle_{\mathrm{C}}\,. \qquad (2.2.9)
\end{aligned}
$$

### 2.2.3 Feynman–Dyson Expansion and Wick's Theorem

In the generating functional $Z[J,\bar{\eta},\eta]$,

$$
\begin{aligned}
Z[J,\bar{\eta},\eta] &= \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}(x)}, \mathrm{i}\frac{\delta}{\delta\eta(x)}\right)\right] \\
&\quad\times Z_0[J]Z_0[\bar{\eta},\eta]\,, \qquad (2.2.8)
\end{aligned}
$$

we expand the exponential on the right-hand side of (2.2.8) into a Taylor series. From (2.2.2) through (2.2.5), we obtain

$$Z[J,\bar{\eta},\eta] = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\mathrm{i}\int \mathrm{d}^4x \mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta \bar{\eta}(x)}, \mathrm{i}\frac{\delta}{\delta \eta(x)}\right)\right]^n$$
$$\times \exp\left[-\mathrm{i}\left(\frac{1}{2}JD_0^{\mathrm{F}}J + \bar{\eta}_\alpha S_{0,\alpha\beta}^{\mathrm{F}}\eta_\beta\right)\right], \tag{2.2.10}$$

which gives the Feynman–Dyson expansion of the connected part of the "full" Green's function (the $(l+m+n)$-point function),

$$G^{\mathrm{C}}_{\alpha_1,\ldots,\alpha_l;1,\ldots,m;\beta_1,\ldots,\beta_n}(x_1,\ldots,x_l;y_1,\ldots,y_m;z_{\beta_1},\ldots,z_{\beta_n}),$$

in terms of $D_0^{\mathrm{F}}(x-y)$, $S_0^{\mathrm{F}}(x-y)$ and the interaction Lagrangian density which is given by $\mathcal{L}_{\mathrm{int}}(\phi(x),\psi(x),\bar{\psi}(x))$. We will find this as follows. In order to produce the $(l+m+n)$-point function, we have the functional derivatives,

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y_1)}\cdots\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y_m)}\frac{1}{\mathrm{i}}\frac{\delta}{\delta \bar{\eta}_{\alpha_1}(x_1)}\cdots\frac{1}{\mathrm{i}}\frac{\delta}{\delta \bar{\eta}_{\alpha_l}(x_l)}$$
$$\times\mathrm{i}\frac{\delta}{\delta \eta_{\beta_n}(z_n)}\cdots\mathrm{i}\frac{\delta}{\delta \eta_{\beta_1}(z_1)}\Big|_{J=\bar{\eta}=\eta=0}, \tag{2.2.11}$$

acting on the left-hand side of (2.2.10), while on the right-hand side of (2.2.10), we obtain the sum of the combinations of $D_0^{\mathrm{F}}(x-y)$, $S_0^{\mathrm{F}}(x-y)$ and $\mathcal{L}_{\mathrm{int}}(\phi(x),\psi(x),\bar{\psi}(x))$ as a result of this operation. We can read off the Feynman rule from $\mathcal{L}_{\mathrm{int}}(\phi(x),\psi(x),\bar{\psi}(x))$ quite easily.

We have Wick's theorem in canonical quantum field theory, which is a trivial consequence of the following identities in the path integral quantization method:

$$\frac{\delta}{\delta J(x)}J(y) = \delta^4(x-y), \tag{2.2.12a}$$

$$\frac{\delta}{\delta \eta_\alpha(x)}\eta_\beta(y) = \delta_{\alpha\beta}\delta^4(x-y), \qquad \frac{\delta}{\delta \bar{\eta}_\alpha(x)}\bar{\eta}_\beta(y) = \delta_{\alpha\beta}\delta^4(x-y). \tag{2.2.12b}$$

Taking the functional derivative of $Z_0[J]Z_0[\bar{\eta},\eta]$ with respect to the external hook, $J(x)$, $\bar{\eta}(x)$ and $\eta(x)$, in the path integral quantization corresponds to taking the contraction in canonical quantum field theory.

## 2.3 Symanzik Construction

Let us summarize the logical structure of the presentation leading to the covariant perturbation theory of Sect. 2.2. We discussed the path integral representation of quantum mechanics in Chap. 1 from the Lagrangian formalism, the Hamiltonian formalism and the Weyl correspondence, gradually making the requisite corrections to the naive Feynman path integral formula obtained in the Lagrangian formalism. In Sect. 2.1, we translated the results of the path integral representation of quantum mechanics to those of quantum field theory. In Sect. 2.2, we discussed covariant perturbation theory from the path integral representation of the generating functional $Z[J,\bar{\eta},\eta]$ of the "full" Green's functions. This procedure is not lengthy compared to the standard canonical procedure, starting from the Tomonaga–Schwinger equation and arriving at covariant perturbation theory. The generating functional $Z[J,\bar{\eta},\eta]$ is, however, not derived from the canonical formalism of quantum field theory directly.

Hence, in this section, we discuss the Symanzik construction which derives the path integral representation of the generating functional $Z[J,\bar{\eta},\eta]$ directly from the canonical formalism of quantum field theory in the Heisenberg picture. In Sect. 2.3.1, we derive the equation of motion of $Z[J,\bar{\eta},\eta]$ from its definition, the Euler–Lagrange equations of motion of the Heisenberg field operators and the equal-time canonical (anti-)commutator. The equation of motion of $Z[J,\bar{\eta},\eta]$ is a functional differential equation. In Sect. 2.3.2, we show that we can integrate this functional differential equation by a functional Fourier transform trivially, and obtain the path integral representation of the generating functional $Z[J,\bar{\eta},\eta]$. In Sect. 2.3.3, we show that the problem of obtaining the generating functional $Z[J,\bar{\eta},\eta]$ is reduced to an external field problem, using the neutral ps-ps coupling as an example. The result of this section validates the translation of Sect. 2.1.

### 2.3.1 Equation of Motion of the Generating Functional

In this section, we consider quantum field theory described by the total Lagrangian density $\mathcal{L}_{\mathrm{tot}}$ of the form,

$$\begin{aligned}\mathcal{L}_{\mathrm{tot}}&\\ &=\frac{1}{4}\left[(\hat{\psi}^{\dagger}(x)\gamma^0)_\alpha, D_{\alpha\beta}(x)\hat{\psi}_\beta(x)\right]+\frac{1}{4}\left[D^{\mathrm{T}}_{\beta\alpha}(-x)(\hat{\psi}^{\dagger}(x)\gamma^0)_\alpha, \hat{\psi}_\beta(x)\right]\\ &\quad+\frac{1}{2}\hat{\phi}(x)K(x)\hat{\phi}(x)+\mathcal{L}_{\mathrm{int}}(\hat{\phi}(x),\hat{\psi}(x),\hat{\psi}^{\dagger}(x)\gamma^0)\,,\end{aligned} \tag{2.3.1}$$

where we have

$$D_{\alpha\beta}(x)=(\mathrm{i}\gamma_\mu\partial^\mu-m+\mathrm{i}\varepsilon)_{\alpha\beta}\,,$$

$$D^{\mathrm{T}}_{\beta\alpha}(-x)=\left(-\mathrm{i}\gamma^{\mathrm{T}}_\mu\partial^\mu-m+\mathrm{i}\varepsilon\right)_{\beta\alpha}\,, \tag{2.3.2a}$$

$$K(x)=-\partial^2-\kappa^2+\mathrm{i}\varepsilon\,, \tag{2.3.2b}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \quad \bar{\psi}_\alpha(x) = (\psi^\dagger(x)\gamma^0)_\alpha \,, \tag{2.3.3}$$

$$I_{\rm tot}\left[\hat{\phi}, \hat{\psi}, \hat{\psi}^\dagger\gamma^0\right] = \int \mathrm{d}^4x \mathcal{L}_{\rm tot}\left((2.3.1)\right), \tag{2.3.4a}$$

$$I_{\rm int}\left[\hat{\phi}, \hat{\psi}, \hat{\psi}^\dagger\gamma^0\right] = \int \mathrm{d}^4x \mathcal{L}_{\rm int}\left(\hat{\phi}(x), \hat{\psi}(x), \hat{\psi}^\dagger(x)\gamma^0\right). \tag{2.3.4b}$$

We have the Euler–Lagrange equations of motion for the field operators:

$$\hat{\psi}_\alpha(x): \quad \frac{\delta \hat{I}_{\rm tot}}{\delta(\hat{\psi}^\dagger(x)\gamma^0)_\alpha} = 0,$$

$$\text{or} \quad D_{\alpha\beta}(x)\hat{\psi}_\beta(x) + \frac{\delta \hat{I}_{\rm int}}{\delta(\hat{\psi}^\dagger(x)\gamma^0)_\alpha} = 0\,, \tag{2.3.5a}$$

$$(\hat{\psi}^\dagger(x)\gamma^0)_\beta: \quad \frac{\delta \hat{I}_{\rm tot}}{\delta\hat{\psi}_\beta(x)} = 0,$$

$$\text{or} \quad -D^{\rm T}_{\beta\alpha}(-x)(\hat{\psi}^\dagger(x)\gamma^0)_\alpha + \frac{\delta \hat{I}_{\rm int}}{\delta\hat{\psi}_\beta(x)} = 0\,, \tag{2.3.5b}$$

$$\hat{\phi}(x): \quad \frac{\delta I_{\rm tot}}{\delta\hat{\phi}(x)} = 0, \quad \text{or} \quad K(x)\hat{\phi}(x) + \frac{\delta \hat{I}_{\rm int}}{\delta\hat{\phi}(x)} = 0\,. \tag{2.3.5c}$$

We have the equal-time canonical (anti-)commutators,

$$\delta(x^0 - y^0)\{\hat{\psi}_\beta(x), (\hat{\psi}^\dagger(y)\gamma^0)_\alpha\} = \gamma^0_{\beta\alpha}\delta^4(x-y)\,, \tag{2.3.6a}$$

$$\begin{aligned}\delta(x^0 - y^0)&\{\hat{\psi}_\beta(x), \hat{\psi}_\alpha(y)\} \\ &= \delta(x^0 - y^0)\{(\hat{\psi}^\dagger(x)\gamma^0)_\beta, (\hat{\psi}^\dagger(y)\gamma^0)_\alpha\} = 0\,,\end{aligned} \tag{2.3.6b}$$

$$\delta(x^0 - y^0)[\hat{\phi}(x), \partial_0^y\hat{\phi}(y)] = \mathrm{i}\delta^4(x-y)\,, \tag{2.3.6c}$$

$$\delta(x^0 - y^0)[\hat{\phi}(x), \hat{\phi}(y)] = \delta(x^0 - y^0)[\partial_0^x\hat{\phi}(x), \partial_0^y\hat{\phi}(y)] = 0\,, \tag{2.3.6d}$$

and the remaining equal-time mixed canonical commutators are equal to 0. We define the generating functional of the "full" Green's functions by

$$
\begin{aligned}
&Z[J,\bar{\eta},\eta] \\
&\quad\equiv \left\langle 0,\text{out}\middle|\mathrm{T}\left(\exp\left[\mathrm{i}\int \mathrm{d}^4x\left\{J(x)\hat{\phi}(x)+\bar{\eta}_\alpha(x)\hat{\psi}_\alpha(x)\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left. +(\hat{\psi}^\dagger(x)\gamma^0)_\beta\eta_\beta(x)\right\}\right]\right)\middle|0,\text{in}\right\rangle \\
&\quad\equiv \sum_{l,m,n=0}^{\infty}\frac{\mathrm{i}^{l+m+n}}{l!m!n!}\langle 0,\text{out}|\mathrm{T}((\bar{\eta}\hat{\psi})^l(J\hat{\phi})^m(\hat{\psi}^\dagger\gamma^0\eta)^n)|0,\text{in}\rangle \\
&\quad\equiv \sum_{l,m,n=0}^{\infty}\frac{\mathrm{i}^{l+m+n}}{l!m!n!}\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_l\mathrm{d}^4y_1\cdots\mathrm{d}^4y_m \\
&\qquad\times\mathrm{d}^4z_1\cdots\mathrm{d}^4z_nJ(y_1)\cdots J(y_m) \\
&\qquad\times\bar{\eta}_{\alpha_l}(x_l)\cdots\bar{\eta}_{\alpha_1}(x_1)\langle 0,\text{out}|\mathrm{T}(\hat{\psi}_{\alpha_1}(x_1)\cdots\hat{\psi}_{\alpha_l}(x_l)\hat{\phi}(y_1)\cdots\hat{\phi}(y_m) \\
&\qquad\times(\hat{\psi}^\dagger(z_1)\gamma^0)_{\beta_1}\cdots(\hat{\psi}^\dagger(z_n)\gamma^0)_{\beta_n})|0,\text{in}\rangle\eta_{\beta_n}(z_n)\cdots\eta_{\beta_1}(z_1) \\
&\quad\equiv \sum_{l,m,n=0}^{\infty}\frac{\mathrm{i}^{l+m+n}}{(l+m+n)!} \\
&\qquad\times\left\langle 0,\text{out}|\mathrm{T}(\bar{\eta}\hat{\psi}+J\hat{\phi}+\hat{\psi}^\dagger\gamma^0\eta)^{l+m+n}|0,\text{in}\right\rangle\,,
\end{aligned}
\tag{2.3.7}
$$

where we have introduced in (2.3.7) the abbreviations

$$
\begin{aligned}
&J\hat{\phi}\equiv\int \mathrm{d}^4yJ(y)\hat{\phi}(y),\quad \bar{\eta}\hat{\psi}\equiv\int \mathrm{d}^4x\bar{\eta}(x)\hat{\psi}(x), \\
&\hat{\psi}^\dagger\gamma^0\eta\equiv\int \mathrm{d}^4z\hat{\psi}^\dagger(z)\gamma^0\eta(z)\,.
\end{aligned}
$$

We observe that

$$
\frac{\delta}{\delta\bar{\eta}_\beta(x)}(\bar{\eta}\hat{\psi})^l = l\hat{\psi}_\beta(x)(\bar{\eta}\hat{\psi})^{l-1}\,, \tag{2.3.8a}
$$

$$
\frac{\delta}{\delta\eta_\alpha(x)}((\hat{\psi}^\dagger\gamma^0)\eta)^n = -n(\hat{\psi}^\dagger(x)\gamma^0)_\alpha((\hat{\psi}^\dagger\gamma^0)\eta)^{n-1}\,, \tag{2.3.8b}
$$

$$
\frac{\delta}{\delta J(x)}(J\hat{\phi})^m = m\hat{\phi}(x)(J\hat{\phi})^{m-1}\,. \tag{2.3.8c}
$$

We also observe from the definition of $Z[J,\bar{\eta},\eta]$,

$$\begin{aligned}
&\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_\beta(x)}Z[J,\bar{\eta},\eta] \\
&\quad=\left\langle 0,\mathrm{out}\left|\mathrm{T}\left(\hat{\psi}_\beta(x)\exp\left[\mathrm{i}\int \mathrm{d}^4z\left\{J(z)\hat{\phi}(z)+\bar{\eta}_\alpha(z)\hat{\psi}_\alpha(z)\right.\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.\left.+(\hat{\psi}^\dagger(z)\gamma^0)_\beta\eta_\beta(z)\right\}\right]\right)\right|0,\mathrm{in}\right\rangle, \qquad (2.3.9\mathrm{a})
\end{aligned}$$

$$\begin{aligned}
&\mathrm{i}\frac{\delta}{\delta\eta_\alpha(x)}Z[J,\bar{\eta},\eta] \\
&\quad=\left\langle 0,\mathrm{out}\left|\mathrm{T}\left((\hat{\psi}^\dagger(x)\gamma^0)_\alpha\exp\left[\mathrm{i}\int \mathrm{d}^4z\left\{J(z)\hat{\phi}(z)+\bar{\eta}_\alpha(z)\hat{\psi}_\alpha(z)\right.\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.\left.+(\hat{\psi}^\dagger(z)\gamma^0)_\beta\eta_\beta(z)\right\}\right]\right)\right|0,\mathrm{in}\right\rangle, \qquad (2.3.9\mathrm{b})
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}Z[J,\bar{\eta},\eta] \\
&\quad=\left\langle 0,\mathrm{out}\left|\mathrm{T}\left(\hat{\phi}(x)\exp\left[\mathrm{i}\int \mathrm{d}^4z\left\{J(z)\hat{\phi}(z)+\bar{\eta}_\alpha(z)\hat{\psi}_\alpha(z)\right.\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.\left.+(\hat{\psi}^\dagger(z)\gamma^0)_\beta\eta_\beta(z)\right\}\right]\right)\right|0,\mathrm{in}\right\rangle. \qquad (2.3.9\mathrm{c})
\end{aligned}$$

From the definition of the time-ordered product and the equal-time canonical (anti-)commutators, (2.3.6a–d), we have at the operator level:

**Fermion**:

$$\begin{aligned}
&D_{\alpha\beta}(x)\mathrm{T}\left(\hat{\psi}_\beta(x)\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}_\alpha\hat{\psi}_\alpha+(\hat{\psi}^\dagger\gamma^0)_\beta\eta_\beta\right\}\right]\right) \\
&\quad=\mathrm{T}\left(D_{\alpha\beta}(x)\hat{\psi}_\beta(x)\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}_\alpha\hat{\psi}_\alpha+(\hat{\psi}^\dagger\gamma^0)_\beta\eta_\beta\right\}\right]\right) \\
&\qquad+\mathrm{T}\left(\mathrm{i}\int \mathrm{d}^4z\cdot\mathrm{i}\gamma^0_{\alpha\beta}\delta(x^0-z^0)\left[\hat{\psi}_\beta(x),(\hat{\psi}^\dagger(z)\gamma^0)_\varepsilon\eta_\varepsilon(z)\right]\right. \\
&\qquad\left.\times\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^\dagger\gamma^0)\eta\right\}\right]\right) \\
&\quad=\mathrm{T}\left(D_{\alpha\beta}(x)\hat{\psi}_\beta(x)\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^\dagger\gamma^0)\eta\right\}\right]\right) \\
&\qquad-\eta_\alpha(x)\mathrm{T}\left(\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^\dagger\gamma^0)\eta\right\}\right]\right), \qquad (2.3.10)
\end{aligned}$$

**Anti-Fermion**:

$$\begin{aligned}
&-D^{\mathrm{T}}_{\beta\alpha}(-x)\mathrm{T}\left((\hat{\psi}^{\dagger}(x)\gamma^0)_\alpha \exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}_\alpha\hat{\psi}_\alpha+(\hat{\psi}^{\dagger}\gamma^0)_\beta\eta_\beta\right\}\right]\right) \\
&\quad=\mathrm{T}\left(-D^{\mathrm{T}}_{\beta\alpha}(-x)(\hat{\psi}^{\dagger}(x)\gamma^0)_\alpha \exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}_\alpha\hat{\psi}_\alpha+(\hat{\psi}^{\dagger}\gamma^0)_\beta\eta_\beta\right\}\right]\right) \\
&\qquad+\mathrm{T}\left(\mathrm{i}\int \mathrm{d}^4z\cdot \mathrm{i}\gamma^0_{\alpha\beta}\delta(x^0-z^0)\left[(\hat{\psi}^{\dagger}(x)\gamma^0)_\alpha,\bar{\eta}_\varepsilon(z)\hat{\psi}_\varepsilon(z)\right]\right. \\
&\qquad\left.\times\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^{\dagger}\gamma^0)\eta\right\}\right]\right) \\
&\quad=\mathrm{T}\left(-D^{\mathrm{T}}_{\beta\alpha}(-x)(\hat{\psi}^{\dagger}(x)\gamma^0)_\alpha \exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^{\dagger}\gamma^0)\eta\right\}\right]\right) \\
&\qquad+\bar{\eta}_\beta(x)\mathrm{T}\left(\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^{\dagger}\gamma^0)\eta\right\}\right]\right), \qquad (2.3.11)
\end{aligned}$$

**Boson**:

$$\begin{aligned}
&K(x)\mathrm{T}\left(\hat{\phi}(x)\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}_\alpha\hat{\psi}_\alpha+(\hat{\psi}^{\dagger}\gamma^0)_\beta\eta_\beta\right\}\right]\right) \\
&\quad=\mathrm{T}\left(K(x)\hat{\phi}(x)\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}_\alpha\hat{\psi}_\alpha+(\hat{\psi}^{\dagger}\gamma^0)_\beta\eta_\beta\right\}\right]\right) \\
&\qquad-\mathrm{T}\left(\mathrm{i}\int \mathrm{d}^4z\cdot\delta(x^0-z^0)[\partial_0\hat{\phi}(x),J(z)\hat{\phi}(z)]\right. \\
&\qquad\left.\times\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^{\dagger}\gamma^0)\eta\right\}\right]\right) \\
&\quad=\mathrm{T}\left(K(x)\hat{\phi}(x)\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^{\dagger}\gamma^0)\eta\right\}\right]\right) \\
&\qquad-J(x)\mathrm{T}\left(\exp\left[\mathrm{i}\left\{J\hat{\phi}+\bar{\eta}\hat{\psi}+(\hat{\psi}^{\dagger}\gamma^0)\eta\right\}\right]\right). \qquad (2.3.12)
\end{aligned}$$

Applying the Euler–Lagrange equations of motion, (2.3.5a–c), to the first terms of the right-hand sides of (2.3.10–2.3.12), and taking the vacuum expectation values of (2.3.10–2.3.12), we obtain the equations of motion of the generating functional $Z[J,\bar{\eta},\eta]$ of the "full" Green's functions as

$$\left\{D_{\alpha\beta}(x)\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_\beta(x)}+\frac{\delta I_{\mathrm{int}}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}},\mathrm{i}\frac{\delta}{\delta\eta}\right]}{\delta\left(\mathrm{i}\frac{\delta}{\delta\eta_\alpha(x)}\right)}+\eta_\alpha(x)\right\}Z[J,\bar{\eta},\eta]=0\,, \qquad (2.3.13a)$$

$$\left\{-D^{\mathrm{T}}_{\beta\alpha}(-x)\mathrm{i}\frac{\delta}{\delta\eta_\alpha(x)}+\frac{\delta I_{\mathrm{int}}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}},\mathrm{i}\frac{\delta}{\delta\eta}\right]}{\delta\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_\beta(x)}\right)}-\bar{\eta}_\beta(x)\right\}Z[J,\bar{\eta},\eta]=0, \qquad (2.3.13b)$$

$$\left\{ K(x)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)} + \frac{\delta I_{\mathrm{int}}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta \bar{\eta}}, \mathrm{i}\frac{\delta}{\delta \eta}\right]}{\delta\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)} + J(x) \right\} Z[J, \bar{\eta}, \eta] = 0\,. \quad (2.3.13\mathrm{c})$$

Equivalently, we can write (2.3.13a–c) with the use of (2.3.5a–c) as

$$\left\{ \frac{\delta I_{\mathrm{tot}}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta \bar{\eta}}, \mathrm{i}\frac{\delta}{\delta \eta}\right]}{\delta\left(\mathrm{i}\frac{\delta}{\delta \eta_\alpha(x)}\right)} + \eta_\alpha(x) \right\} Z[J, \bar{\eta}, \eta] = 0\,, \qquad (2.3.14\mathrm{a})$$

$$\left\{ \frac{\delta I_{\mathrm{tot}}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta \bar{\eta}}, \mathrm{i}\frac{\delta}{\delta \eta}\right]}{\delta\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta \bar{\eta}_\beta(x)}\right)} - \bar{\eta}_\beta(x) \right\} Z[J, \bar{\eta}, \eta] = 0\,, \qquad (2.3.14\mathrm{b})$$

$$\left\{ \frac{\delta I_{\mathrm{tot}}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta \bar{\eta}}, \mathrm{i}\frac{\delta}{\delta \eta}\right]}{\delta\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)} + J(x) \right\} Z[J, \bar{\eta}, \eta] = 0\,. \qquad (2.3.14\mathrm{c})$$

We note that the coefficients of the external hook terms, $\eta_\alpha(x)$, $\bar{\eta}_\beta(x)$ and $J(x)$, in (2.3.14a), (2.3.14b) and (2.3.14c) are $\pm 1$, which is a reflection of the fact that we are dealing with canonical quantum field theory and originates from the equal-time canonical (anti-)commutators.

### 2.3.2 Method of the Functional Fourier Transform

We define the functional Fourier transform $\tilde{Z}[\phi, \psi, \bar{\psi}]$ of the generating functional $Z[J, \bar{\eta}, \eta]$ by

$$Z[J, \bar{\eta}, \eta] \equiv \int \mathcal{D}[\phi]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\tilde{Z}[\phi, \psi, \bar{\psi}] \exp\left[\mathrm{i}(J\phi + \bar{\eta}\psi + \bar{\psi}\eta)\right]. \qquad (2.3.15)$$

By functional integration by parts, we obtain the identity

$$\begin{bmatrix} \eta_\alpha(x) \\ -\bar{\eta}_\beta(x) \\ J(x) \end{bmatrix} Z[J, \bar{\eta}, \eta]$$

$$= \int \mathcal{D}[\phi]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\tilde{Z}[\phi, \psi, \bar{\psi}] \begin{bmatrix} \dfrac{1}{\mathrm{i}}\dfrac{\delta}{\delta \bar{\psi}_\alpha(x)} \\ \dfrac{1}{\mathrm{i}}\dfrac{\delta}{\delta \psi_\beta(x)} \\ \dfrac{1}{\mathrm{i}}\dfrac{\delta}{\delta \phi(x)} \end{bmatrix}$$

$$\times \exp[\mathrm{i}(J\phi + \bar{\eta}\psi + \bar{\psi}\eta)]$$

$$= \int \mathcal{D}[\phi]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \left\{ \begin{bmatrix} -\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\psi}_\alpha(x)} \\ -\frac{1}{\mathrm{i}}\frac{\delta}{\delta\psi_\beta(x)} \\ -\frac{1}{\mathrm{i}}\frac{\delta}{\delta\phi(x)} \end{bmatrix} \tilde{Z}[\phi,\psi,\bar{\psi}] \right\}$$

$$\times \exp\left[\mathrm{i}(J\phi + \bar{\eta}\psi + \bar{\psi}\eta)\right] . \tag{2.3.16}$$

With the identity, (2.3.16), we have the equations of motion of the functional Fourier transform $\tilde{Z}[\phi,\psi,\bar{\psi}]$ of the generating functional $Z[J,\bar{\eta},\eta]$ from (2.3.14a–c) as

$$\left\{ \frac{\delta I_{\mathrm{tot}}[\phi,\psi,\bar{\psi}]}{\delta\bar{\psi}_\alpha(x)} - \frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\psi}_\alpha(x)} \right\} \tilde{Z}[\phi,\psi,\bar{\psi}] = 0 , \tag{2.3.17a}$$

$$\left\{ \frac{\delta I_{\mathrm{tot}}[\phi,\psi,\bar{\psi}]}{\delta\psi_\beta(x)} - \frac{1}{\mathrm{i}}\frac{\delta}{\delta\psi_\beta(x)} \right\} \tilde{Z}[\phi,\psi,\bar{\psi}] = 0 , \tag{2.3.17b}$$

$$\left\{ \frac{\delta I_{\mathrm{tot}}[\phi,\psi,\bar{\psi}]}{\delta\phi(x)} - \frac{1}{\mathrm{i}}\frac{\delta}{\delta\phi(x)} \right\} \tilde{Z}[\phi,\psi,\bar{\psi}] = 0 . \tag{2.3.17c}$$

We divide (2.3.17a–c) by $\tilde{Z}[\phi,\psi,\bar{\psi}]$, and obtain

$$\frac{\delta}{\delta\bar{\psi}_\alpha(x)} \ln \tilde{Z}[\phi,\psi,\bar{\psi}] = \mathrm{i}\frac{\delta}{\delta\bar{\psi}_\alpha(x)} I_{\mathrm{tot}}[\phi,\psi,\bar{\psi}] , \tag{2.3.18a}$$

$$\frac{\delta}{\delta\psi_\beta(x)} \ln \tilde{Z}[\phi,\psi,\bar{\psi}] = \mathrm{i}\frac{\delta}{\delta\psi_\beta(x)} I_{\mathrm{tot}}[\phi,\psi,\bar{\psi}] , \tag{2.3.18b}$$

$$\frac{\delta}{\delta\phi(x)} \ln \tilde{Z}[\phi,\psi,\bar{\psi}] = \mathrm{i}\frac{\delta}{\delta\phi(x)} I_{\mathrm{tot}}[\phi,\psi,\bar{\psi}] . \tag{2.3.18c}$$

We can immediately integrate (2.3.18a–c) with the result,

$$\begin{aligned} \tilde{Z}[\phi,\psi,\bar{\psi}] &= \frac{1}{C_V} \exp[\mathrm{i} I_{\mathrm{tot}}[\phi,\psi,\bar{\psi}]] \\ &= \frac{1}{C_V} \exp\left[\mathrm{i}\int \mathrm{d}^4 z \mathcal{L}_{\mathrm{tot}}(\phi(z),\psi(z),\bar{\psi}(z))\right] . \end{aligned} \tag{2.3.19}$$

From (2.3.19), we obtain the generating functional $Z[J,\bar{\eta},\eta]$ in the functional integral representation,

$$Z[J,\bar{\eta},\eta] = \frac{1}{C_V}\int \mathcal{D}[\phi]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\exp\left[\mathrm{i}\int \mathrm{d}^4z\,\{\mathcal{L}_{\text{tot}}(\phi(z),\psi(z),\bar{\psi}(z)) + J(z)\phi(z) + \bar{\eta}(z)\psi(z) + \bar{\psi}(z)\eta(z)\}\right]. \tag{2.3.20}$$

Here, $C_V$ is a normalization constant. If we normalize $Z[J,\bar{\eta},\eta]$ as

$$Z[J=\bar{\eta}=\eta=0]=1\,, \tag{2.3.21}$$

we find $C_V$ to be the vacuum-to-vacuum transition amplitude,

$$C_V = \int \mathcal{D}[\phi]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\exp[\mathrm{i}I_{\text{tot}}[\phi,\psi,\bar{\psi}]] = \langle 0,\text{out}|0,\text{in}\rangle\,. \tag{2.3.22}$$

Another normalization we may use the following,

$$Z[J=\bar{\eta}=\eta=0] = \langle 0,\text{out}|0,\text{in}\rangle\,, \tag{2.3.23a}$$

with

$$C_V = 1\,. \tag{2.3.23b}$$

### 2.3.3 External Field Problem

In this section, we consider the neutral ps-ps coupling

$$\mathcal{L}_{\text{int}}(\phi(x),\psi(x),\bar{\psi}(x)) = g_0\bar{\psi}_\alpha(x)(\gamma_5)_{\alpha\beta}\psi_\beta(x)\phi(x) \tag{2.3.24}$$

as an example, and demonstrate that the problem of the interacting system is reduced to finding the Green's functions of the external field problem with slight modification of the path integral representation, (2.3.20), of the generating functional $Z[J,\bar{\eta},\eta]$.

To begin with, we separate the exponent of (2.3.20) into a (pseudo) scalar part and a fermionic part,

$$\begin{aligned}&\mathcal{L}_{\text{tot}} + J\phi + \bar{\eta}\psi + \bar{\psi}\eta \\ &\quad= \left\{\tfrac{1}{2}\phi K\phi + J\phi\right\} + \left\{\bar{\psi}_\alpha(D_{\alpha\beta} + g_0(\gamma_5)_{\alpha\beta}\phi)\psi_\beta + \bar{\eta}_\beta\psi_\beta + \bar{\psi}_\alpha\eta_\alpha\right\}.\end{aligned} \tag{2.3.25}$$

We define $f[\phi,\bar{\eta},\eta]$ by

$$f[\phi,\bar{\eta},\eta] \equiv \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\exp\left[\mathrm{i}\int \mathrm{d}^4z\{\bar{\psi}_\alpha(z)(D_{\alpha\beta}(z) + g_0(\gamma_5)_{\alpha\beta}\phi(z))\psi_\beta(z) + \bar{\eta}_\beta(z)\psi_\beta(z) + \bar{\psi}_\alpha(z)\eta_\alpha(z)\}\right]. \tag{2.3.26}$$

Then, we have

$$\begin{aligned}
Z[J,\bar{\eta},\eta] &= \frac{1}{C_V}\int \mathcal{D}[\phi] f[\phi,\bar{\eta},\eta] \\
&\quad\times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\left\{\tfrac{1}{2}\phi(z)K(z)\phi(z)+J(z)\phi(z)\right\}\right] \\
&= \frac{1}{C_V} f\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J},\bar{\eta},\eta\right]\int \mathcal{D}[\phi]\exp\left[\mathrm{i}\left(\tfrac{1}{2}\phi K\phi + J\phi\right)\right] \\
&= \frac{1}{C_V'} f\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J},\bar{\eta},\eta\right]\exp\left[-\frac{\mathrm{i}}{2}JD_0^{\mathrm{F}}J\right] \\
&= \frac{1}{C_V'}\int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \\
&\quad\times\exp\left[\mathrm{i}\int \mathrm{d}^4 z\left\{\bar{\psi}_\alpha(z)\left(D_{\alpha\beta}(z)+g_0(\gamma_5)_{\alpha\beta}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right)\psi_\beta(z)\right.\right. \\
&\quad\left.\left.+\bar{\eta}_\beta(z)\psi_\beta(z)+\bar{\psi}_\alpha(z)\eta_\alpha(z)\right\}\right]\exp\left[-\frac{\mathrm{i}}{2}JD_0^{\mathrm{F}}J\right] \\
&\equiv \frac{1}{C_V'}\int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\exp\left[\mathrm{i}\int \mathrm{d}^4 z\left\{\bar{\psi}_\alpha(z)\left(S_{\mathrm{F}}^{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}}(z)\right)_{\alpha\beta}\psi_\beta(z)\right.\right. \\
&\quad\left.\left.+\bar{\eta}_\beta(z)\psi_\beta(z)+\bar{\psi}_\alpha(z)\eta_\alpha(z)\right\}\right]\exp\left[-\frac{\mathrm{i}}{2}JD_0^{\mathrm{F}}J\right], \qquad (2.3.27)
\end{aligned}$$

where $S_{\mathrm{F}}^{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}}(x)$ is the fermion Green's function in the presence of an external field and is defined by

$$\begin{aligned}
S_{\mathrm{F}}^{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}}(x)_{\alpha\beta} &\equiv \left(D(x)+g_0\gamma_5\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)^{-1}_{\alpha\beta} \\
&= \left(1+g_0 S_0^{\mathrm{F}}(x)\gamma_5\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)^{-1}_{\alpha\varepsilon} S_0^{\mathrm{F}}(x)_{\varepsilon\beta}\,. \qquad (2.3.28)
\end{aligned}$$

Next, we perform a function change of the fermion variable,

$$\psi'(x) = \left(S_{\mathrm{F}}^{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}}(x)\right)^{-1/2}\psi(x), \quad \mathcal{D}[\psi']\mathcal{D}[\bar{\psi}'] = \mathrm{Det}\left(S_{\mathrm{F}}^{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}}\right)\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\,. \qquad (2.3.29)$$

As a result of fermion number integration, we obtain

$$Z[J,\bar{\eta},\eta] = \frac{1}{C_V'}\left\{\mathrm{Det}\left(S_{\mathrm{F}}^{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}}\right)\right\}^{-1}\exp\left[-\mathrm{i}\bar{\eta}_\alpha S_{\mathrm{F},\alpha\beta}^{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}}\eta_\beta\right]\exp\left[-\frac{\mathrm{i}}{2}JD_0^{\mathrm{F}}J\right]$$

$$= \frac{1}{C''_V} \left\{ \mathrm{Det} \left( 1 + g_0 S_0^{\mathrm{F}} \gamma_5 \frac{1}{\mathrm{i}} \frac{\delta}{\delta J} \right) \right\}^{-1}$$

$$\times \exp \left[ -\mathrm{i} \bar{\eta}_\alpha S_{\mathrm{F},\alpha\beta}^{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}} \eta_\beta \right] \exp \left[ -\frac{\mathrm{i}}{2} J D_0^{\mathrm{F}} J \right], \tag{2.3.30}$$

where the constants, $(\mathrm{Det}\, D_0^{\mathrm{F}})^{-1/2}$ and $(\mathrm{Det}\, S_0^{\mathrm{F}})^{-1}$, are absorbed into the normalization constants, $C'_V$ and $C''_V$, with $C''_V$ given by

$$C''_V = \left\{ \mathrm{Det} \left( 1 + g_0 S_0^{\mathrm{F}} \gamma_5 \frac{1}{\mathrm{i}} \frac{\delta}{\delta J} \right) \right\}^{-1} \exp \left[ -\frac{\mathrm{i}}{2} J D_0^{\mathrm{F}} J \right] \Bigg|_{J=0}. \tag{2.3.31}$$

As is obvious from (2.3.30) that, once we know $S_{\mathrm{F}}^{\phi}(x)$, i.e., once the external field problem is solved, we have solved the problem of the interacting system as well.

Alternatively, using the translation property of

$$\exp \left[ \int \mathrm{d}^4 x h(x) \frac{\delta}{\delta J(x)} \right],$$

with respect to $J(x)$, we can rewrite (2.3.20) as

$$\begin{aligned} &Z[J, \bar{\eta}, \eta] \\ &= \frac{1}{C_V} \exp \left[ \mathrm{i} \int \mathrm{d}^4 z \mathcal{L}_{\mathrm{int}} \left( \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(z)}, \frac{1}{\mathrm{i}} \frac{\delta}{\delta \bar{\eta}(z)}, \mathrm{i} \frac{\delta}{\delta \eta(z)} \right) \right] Z_0[J] Z_0[\bar{\eta}, \eta] \\ &= \frac{1}{C_V} \exp \left[ \int \mathrm{d}^4 z g_0 \frac{\delta}{\delta \eta(z)} \gamma_5 \frac{\delta}{\delta \bar{\eta}(z)} \frac{\delta}{\delta J(z)} \right] \\ &\quad \times \exp \left[ -\frac{\mathrm{i}}{2} J D_0^{\mathrm{F}} J \right] \exp[-\mathrm{i} \bar{\eta} S_0^{\mathrm{F}} \eta] \\ &= \frac{1}{C_V} \exp \left[ -\frac{\mathrm{i}}{2} \iint \mathrm{d}^4 x \mathrm{d}^4 y \left( J(x) + g_0 \frac{\delta}{\delta \eta(x)} \gamma_5 \frac{\delta}{\delta \bar{\eta}(x)} \right) D_0^{\mathrm{F}}(x - y) \right. \\ &\quad \left. \times \left( J(y) + g_0 \frac{\delta}{\delta \eta(y)} \gamma_5 \frac{\delta}{\delta \bar{\eta}(y)} \right) \right] \exp \left[ -\mathrm{i} \bar{\eta} S_0^{\mathrm{F}} \eta \right], \end{aligned} \tag{2.3.32}$$

with the normalization constant $C_V$ given by

$$C_V = \exp \left[ -\frac{\mathrm{i}}{2} g_0^2 \frac{\delta}{\delta \eta} \gamma_5 \frac{\delta}{\delta \bar{\eta}} D_0^{\mathrm{F}} \frac{\delta}{\delta \eta} \gamma_5 \frac{\delta}{\delta \bar{\eta}} \right] \exp \left[ -\mathrm{i} \bar{\eta} S_0^{\mathrm{F}} \eta \right] |_{\bar{\eta}=\eta=0}. \tag{2.3.33}$$

In this alternative approach, we need not know the solution, $S_{\mathrm{F}}^{\phi}(x)$, to the external field problem.

Either of these methods works as long as the interaction Lagrangian density $\mathcal{L}_{\mathrm{int}}(\phi(x), \psi(x), \bar{\psi}(x))$ is given by the Yukawa coupling.

## 2.4 Schwinger Theory of the Green's Function

The "full" Green's functions introduced in Sects. 2.1 and 2.2 are entirely different from the Green's functions of linear differential equations encountered in mathematical physics. Due to the presence of the interaction Lagrangian density $\mathcal{L}_{\text{int}}(\phi(x), \psi(x), \bar{\psi}(x))$, the equation of motion of the $n$-point "full" Green's function involves higher-order "full" Green's functions. The system of equations of motion is nonlinear and actually forms an infinite system. In Sect. 2.2, we defined the "full" Green's functions in an essentially perturbative manner and did not know their equations of motion entirely. In this section, in order to remedy these deficiencies, we discuss the Schwinger–Dyson equation.

In Sect. 2.4.1, we derive a coupled system of equations of motion of the "full" Green's functions using the equation of motion of $Z[J, \bar{\eta}, \eta]$, (2.3.13a–c), as the starting point. In Sect. 2.4.2, with the introduction of the proper self-energy parts and vertex operators after Dyson, we decouple the system of equations of motion and derive the Schwinger–Dyson equation which is exact and closed. From this equation, we can derive covariant perturbation theory by iteration just like Sect. 2.2. On top of this, based on the Schwinger–Dyson equation, we can discuss the nonperturbative behavior of QED and QCD with the tri-$\Gamma$ approximation.

In the discussion of this section, we do not use the path integral representation of $Z[J, \bar{\eta}, \eta]$. Instead, we start from the equations of motion of $Z[J, \bar{\eta}, \eta]$, (2.3.13a–c). Furthermore, contrary to the Feynman–Dyson expansion of Sect. 2.2, we can develop covariant perturbation theory based on the exact equation of motion in closed form. To this end, we discuss the Schwinger theory of the Green's function in this section.

### 2.4.1 Definition of the Green's Function and the Equation of Motion

We discuss the Schwinger theory of the Green's function with the interaction Lagrangian density $\mathcal{L}_{\text{int}}(\hat{\phi}(x), \hat{\psi}(x), \hat{\psi}^{\dagger}(x)\gamma^0)$ of the Yukawa coupling in mind,

$$\mathcal{L}_{\text{int}}(\hat{\phi}(x), \hat{\psi}(x), \hat{\psi}^{\dagger}(x)\gamma^0) = G_0(\hat{\psi}^{\dagger}(x)\gamma^0)_{\alpha}\gamma_{\alpha\beta}(x)\hat{\psi}_{\beta}(x)\hat{\phi}(x)\,, \tag{2.4.1}$$

with

$$G_0 = \begin{cases} g_0 \\ f \\ e, \end{cases} \quad \gamma(x) = \begin{cases} \gamma_5 \\ \gamma_5\tau_i \\ \gamma_\mu, \end{cases} \quad \hat{\psi}(x) = \begin{cases} \hat{\psi}_\alpha(x) \\ \hat{\psi}_{N,\alpha}(x) \\ \hat{\psi}_\alpha(x), \end{cases} \quad \phi(x) = \begin{cases} \hat{\phi}(x) \\ \hat{\phi}_i(x) \\ \hat{A}_\mu(x). \end{cases} \tag{2.4.2}$$

We define the vacuum expectation values, $\langle F \rangle^{J,\bar{\eta},\eta}$ and $\langle F \rangle^{J}$, of the operator function $F(\hat{\phi}(x), \hat{\psi}(x), \hat{\psi}^{\dagger}(x)\gamma^0)$ in the presence of the external hook terms $\{J, \bar{\eta}, \eta\}$ by

$$\langle F\rangle^{J,\bar{\eta},\eta} \equiv \frac{1}{Z[J,\bar{\eta},\eta]} F\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}(x)}, \mathrm{i}\frac{\delta}{\delta\eta(x)}\right) Z[J,\bar{\eta},\eta]\,, \tag{2.4.3a}$$

$$\langle F\rangle^{J} = \langle F\rangle^{J,\bar{\eta},\eta}|_{\bar{\eta}=\eta=0}\,. \tag{2.4.3b}$$

We define the connected parts of the two-point "full" Green's functions in the presence of the external hook $J(x)$ by

**Fermion**:

$$\begin{aligned}
S'^{J}_{\mathrm{F},\alpha\beta}(x_1,x_2) &\equiv \frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_\alpha(x_1)}\mathrm{i}\frac{\delta}{\delta\eta_\beta(x_2)}\frac{1}{\mathrm{i}}\ln Z[J,\bar{\eta},\eta]\Big|_{\bar{\eta}=\eta=0} \\
&= \frac{1}{\mathrm{i}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_\alpha(x_1)}\right)\langle(\hat{\psi}^\dagger(x_2)\gamma^0)_\beta\rangle^{J,\bar{\eta},\eta}\Big|_{\bar{\eta}=\eta=0} \\
&= \frac{1}{\mathrm{i}}\Big\{\langle\hat{\psi}_\alpha(x_1)(\hat{\psi}^\dagger(x_2)\gamma^0)_\beta\rangle^{J,\bar{\eta},\eta}\Big|_{\bar{\eta}=\eta=0} \\
&\quad -\langle\hat{\psi}_\alpha(x_1)\rangle^{J,\bar{\eta},\eta}\langle(\hat{\psi}^\dagger(x_2)\gamma^0)_\beta\rangle^{J,\bar{\eta},\eta}\Big|_{\bar{\eta}=\eta=0}\Big\} \\
&= \frac{1}{\mathrm{i}}\langle\hat{\psi}_\alpha(x_1)(\hat{\psi}^\dagger(x_2)\gamma^0)_\beta\rangle^{J} \\
&\equiv \frac{1}{\mathrm{i}}\langle 0,\mathrm{out}|\mathrm{T}(\hat{\psi}_\alpha(x_1)(\hat{\psi}^\dagger(x_2)\gamma^0)_\beta)|0,\mathrm{in}\rangle^{J}_{\mathrm{C}}\,,
\end{aligned} \tag{2.4.4}$$

where we have,

$$\langle\hat{\psi}_\alpha(x_1)\rangle^{J,\bar{\eta},\eta}\Big|_{\bar{\eta}=\eta=0} = \langle(\hat{\psi}^\dagger(x_2)\gamma^0)_\beta\rangle^{J,\bar{\eta},\eta}\Big|_{\bar{\eta}=\eta=0} = 0\,, \tag{2.4.5}$$

and

**Boson**:

$$\begin{aligned}
D'^{J}_{\mathrm{F}}(x_1,x_2) &\equiv \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_1)}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_2)}\frac{1}{\mathrm{i}}\ln Z[J,\bar{\eta},\eta]\Big|_{\bar{\eta}=\eta=0} \\
&= \frac{1}{\mathrm{i}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_1)}\langle\hat{\phi}(x_2)\rangle^{J}\right) \\
&= \frac{1}{\mathrm{i}}\{\langle\hat{\phi}(x_1)\hat{\phi}(x_2)\rangle^{J} - \langle\hat{\phi}(x_1)\rangle^{J}\langle\hat{\phi}(x_2)\rangle^{J}\} \\
&\equiv \frac{1}{\mathrm{i}}\langle 0,\mathrm{out}|\mathrm{T}(\hat{\phi}(x_1)\hat{\phi}(x_2))|0,\mathrm{in}\rangle^{J}_{\mathrm{C}}\,,
\end{aligned} \tag{2.4.6}$$

where we assume that

$$\langle\hat{\phi}(x)\rangle^{J}|_{J=0} = 0\,. \tag{2.4.7}$$

We have the equations of motion of $Z[J,\bar{\eta},\eta]$, (2.3.13a–c), when the interaction Lagrangian density $\mathcal{L}_{\mathrm{int}}(\hat{\phi}(x),\hat{\psi}(x),\hat{\psi}^\dagger(x)\gamma^0)$ is given by (2.4.1):

$$\left\{ D_{\alpha\beta}(x) \left( \frac{1}{\mathrm{i}} \frac{\delta}{\delta \bar{\eta}_{\beta}(x)} \right) + G_0 \gamma_{\alpha\beta}(x) \left( \frac{1}{\mathrm{i}} \frac{\delta}{\delta \bar{\eta}_{\beta}(x)} \right) \left( \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)} \right) \right\} Z[J, \bar{\eta}, \eta]$$
$$= -\eta_{\alpha}(x) Z[J, \bar{\eta}, \eta] , \tag{2.4.8a}$$

$$\left\{ -D^{\mathrm{T}}_{\beta\alpha}(-x) \left( \mathrm{i} \frac{\delta}{\delta \eta_{\alpha}(x)} \right) - G_0 \left( \mathrm{i} \frac{\delta}{\delta \eta_{\alpha}(x)} \right) \gamma_{\alpha\beta}(x) \left( \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)} \right) \right\} Z[J, \bar{\eta}, \eta]$$
$$= +\bar{\eta}_{\beta}(x) Z[J, \bar{\eta}, \eta] , \tag{2.4.8b}$$

$$\left\{ K(x) \left( \frac{1}{\mathrm{i}} \frac{\delta}{\delta J J(x)} \right) + G_0 \left( \mathrm{i} \frac{\delta}{\delta \eta_{\alpha}(x)} \right) \gamma_{\alpha\beta}(x) \left( \frac{1}{\mathrm{i}} \frac{\delta}{\delta \bar{\eta}_{\beta}(x)} \right) \right\} Z[J, \bar{\eta}, \eta]$$
$$= -J(x) Z[J, \bar{\eta}, \eta] . \tag{2.4.8c}$$

Dividing (2.4.8a–c) by $Z[J, \bar{\eta}, \eta]$, and referring to (2.4.3), we obtain the equations of motion for

$$\langle \hat{\psi}_{\beta}(x) \rangle^{J,\bar{\eta},\eta}, \ \langle (\hat{\psi}^{\dagger}(x)\gamma^0)_{\alpha} \rangle^{J,\bar{\eta},\eta} \text{ and } \langle \hat{\phi}(x) \rangle^{J,\bar{\eta},\eta}$$

as

$$D_{\alpha\beta}(x) \langle \hat{\psi}_{\beta}(x) \rangle^{J,\bar{\eta},\eta} + G_0 \gamma_{\alpha\beta}(x) \langle \hat{\psi}_{\beta}(x) \hat{\phi}(x) \rangle^{J,\bar{\eta},\eta} = -\eta_{\alpha}(x) , \tag{2.4.9}$$

$$-D^{\mathrm{T}}_{\beta\alpha}(-x) \langle (\hat{\psi}^{\dagger}(x)\gamma^0)_{\alpha} \rangle - G_0 \gamma_{\alpha\beta}(x) \langle (\hat{\psi}^{\dagger}(x)\gamma^0)_{\alpha} \hat{\phi}(x) \rangle^{J,\bar{\eta},\eta} = +\bar{\eta}_{\beta}(x) , \tag{2.4.10}$$

$$K(x) \langle \hat{\phi}(x) \rangle^{J,\bar{\eta},\eta} + G_0 \gamma_{\alpha\beta}(x) \langle (\hat{\psi}^{\dagger}(x)\gamma^0)_{\alpha} \hat{\psi}_{\beta}(x) \rangle^{J,\bar{\eta},\eta} = -J(x) . \tag{2.4.11}$$

We take the following functional derivatives.

$$\mathrm{i} \frac{\delta}{\delta \eta_{\varepsilon}(y)} (2.4.9) \Big|_{\bar{\eta}=\eta=0} :$$
$$D_{\alpha\beta}(x) \langle (\hat{\psi}^{\dagger}(y)\gamma^0)_{\varepsilon} \hat{\psi}_{\beta}(x) \rangle^J$$
$$+ G_0 \gamma_{\alpha\beta}(x) \langle (\hat{\psi}^{\dagger}(y)\gamma^0)_{\varepsilon} \hat{\psi}_{\beta}(x) \hat{\phi}(x) \rangle^J = -\mathrm{i} \delta_{\alpha\varepsilon} \delta^4(x-y) ,$$

$$\frac{1}{\mathrm{i}} \frac{\delta}{\delta \bar{\eta}_{\varepsilon}(y)} (2.4.10) \Big|_{\bar{\eta}=\eta=0} :$$
$$-D^{\mathrm{T}}_{\beta\alpha}(-x) \langle \hat{\psi}_{\varepsilon}(y) (\hat{\psi}^{\dagger}(x)\gamma^0)_{\alpha} \rangle^J$$
$$- G_0 \gamma_{\alpha\beta}(x) \langle \hat{\psi}_{\varepsilon}(y) (\hat{\psi}^{\dagger}(x)\gamma^0)_{\alpha} \hat{\phi}(x) \rangle^J = -\mathrm{i} \delta_{\beta\varepsilon} \delta^4(x-y) ,$$

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}(2.4.11)\Big|_{\bar{\eta}=\eta=0}:$$

$$K(x)\{\langle\hat{\phi}(y)\hat{\phi}(x)\rangle^J - \langle\hat{\phi}(y)\rangle^J\langle\hat{\phi}(x)\rangle^J\} + G_0\gamma_{\alpha\beta}(x)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\langle(\hat{\psi}^\dagger(x)\gamma^0)_\alpha\hat{\psi}_\beta(x)\rangle^J = \mathrm{i}\delta^4(x-y)\,.$$

These equations are part of an infinite system of coupled equations. We observe the following identities.

$$\langle(\hat{\psi}^\dagger(y)\gamma^0)_\varepsilon\hat{\psi}_\beta(x)\rangle^J = -\mathrm{i}S'^J_{\mathrm{F},\beta\varepsilon}(x,y)\,,$$

$$\langle\hat{\psi}_\varepsilon(y)(\hat{\psi}^\dagger(x)\gamma^0)_\alpha\rangle^J = \mathrm{i}S'^J_{\mathrm{F},\varepsilon\alpha}(y,x)\,,$$

$$\begin{aligned}\langle(\hat{\psi}^\dagger(y)\gamma^0)_\varepsilon\hat{\psi}_\beta(x)\hat{\phi}(x)\rangle^J &= \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\langle(\hat{\psi}^\dagger(y)\gamma^0)_\varepsilon\hat{\psi}_\beta(x)\rangle^J \\ &\quad+\langle(\hat{\psi}^\dagger(y)\gamma^0)_\varepsilon\hat{\psi}_\beta(x)\rangle^J\langle\hat{\phi}(x)\rangle^J \\ &= -\mathrm{i}\left(\langle\hat{\phi}(x)\rangle^J + \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)S'^J_{\mathrm{F},\beta\varepsilon}(x,y)\,,\end{aligned}$$

$$\begin{aligned}\langle\hat{\psi}_\varepsilon(y)(\hat{\psi}^\dagger(x)\gamma^0)_\alpha\hat{\phi}(x)\rangle^J &= \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\langle\hat{\psi}_\varepsilon(y)(\hat{\psi}^\dagger(x)\gamma^0)_\alpha\rangle^J \\ &\quad+\langle\hat{\psi}_\varepsilon(y)(\hat{\psi}^\dagger(x)\gamma^0)_\alpha\rangle^J\langle\hat{\phi}(x)\rangle^J \\ &= \mathrm{i}\left(\langle\hat{\phi}(x)\rangle^J + \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)S'^J_{\mathrm{F},\varepsilon\alpha}(y,x)\,.\end{aligned}$$

With these identities, we obtain the equations of motion of the connected parts of the two-point "full" Green's functions in the presence of the external hook $J(x)$ as

$$\left\{D_{\alpha\beta}(x) + G_0\gamma_{\alpha\beta}(x)\left(\langle\hat{\phi}(x)\rangle^J + \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right\}S'^J_{\mathrm{F},\beta\varepsilon}(x,y) = \delta_{\alpha\varepsilon}\delta^4(x-y)\,, \tag{2.4.12a}$$

$$\left\{D^{\mathrm{T}}_{\beta\alpha}(-x) + G_0\gamma_{\alpha\beta}(x)\left(\langle\hat{\phi}(x)\rangle^J + \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right\}S'^J_{\mathrm{F},\varepsilon\alpha}(y,x) = \delta_{\beta\varepsilon}\delta^4(x-y)\,, \tag{2.4.13a}$$

$$K(x)D_{\mathrm{F}}^{\prime J}(x,y)-G_0\gamma_{\alpha\beta}(x)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}S_{\mathrm{F},\beta\alpha}^{\prime J}(x,x_{\pm})=\delta^4(x-y)\,. \tag{2.4.14a}$$

Since the transpose of (2.4.13a) is (2.4.12a), we only have to consider (2.4.12a) and (2.4.14a). At first sight, we may get the impression that we have the equations of motion of the two-point "full" Green's functions, $S_{\mathrm{F},\alpha\beta}^{\prime J}(x,y)$ and $D_{\mathrm{F}}^{\prime J}(x,y)$ in closed form. Due to the presence of the functional derivative,

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)},$$

however, (2.4.12a), (2.4.13a) and (2.4.14a) involve three-point "full" Green's functions and are merely part of an infinite system of coupled nonlinear equations of motion of the "full" Green's functions.

### 2.4.2 Proper Self-Energy Parts and the Vertex Operator

In this section, we use the variables, "1", "2", "3", ... to represent the space-time indices, $t$, $x$, $y$, $z$, the spinor indices, $\alpha$, $\beta$, $\gamma$, ..., as well as other internal indices, $i$, $j$, $k$, ....

With the use of the "free" Green's functions, $S_0^{\mathrm{F}}(1-2)$ and $D_0^{\mathrm{F}}(1-2)$, defined by

$$D(1)S_0^{\mathrm{F}}(1-2)=1\,, \tag{2.4.15a}$$

$$K(1)D_0^{\mathrm{F}}(1-2)=1\,, \tag{2.4.15b}$$

we rewrite the functional differential equations satisfied by the "full" Green's functions $S_{\mathrm{F}}^{\prime J}(1,2)$ and $D_{\mathrm{F}}^{\prime J}(1,2)$, (2.4.12) and (2.4.14), as the integral equations

$$\begin{aligned}S_{\mathrm{F}}^{\prime J}(1,2)=S_0^{\mathrm{F}}(1-2)&+S_0^{\mathrm{F}}(1-3)(-G_0\gamma(3))\\&\times\left(\langle\hat{\phi}(3)\rangle^J+\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}\right)S_{\mathrm{F}}^{\prime J}(3,2)\,,\end{aligned} \tag{2.4.12b}$$

$$\begin{aligned}D_{\mathrm{F}}^{\prime J}(1,2)=D_0^{\mathrm{F}}(1-2)&+D_0^{\mathrm{F}}(1-3)\\&\times\left(G_0\mathrm{tr}\gamma(3)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(2)}S_{\mathrm{F}}^{\prime J}(3,3_{\pm})\right).\end{aligned} \tag{2.4.14b}$$

We compare (2.4.12b) and (2.4.14b) with the defining integral equations of the proper self-energy parts, $\boldsymbol{\Sigma}^*$ and $\boldsymbol{\Pi}^*$, due to Dyson, in the presence of the external hook $J(x)$,

$$\begin{aligned}S_{\mathrm{F}}^{\prime J}(1,2)=S_0^{\mathrm{F}}(1-2)&+S_0^{\mathrm{F}}(1-3)(-G_0\gamma(3)\langle\phi(3)\rangle^J)S_{\mathrm{F}}^{\prime J}(3,2)\\&+S_0^{\mathrm{F}}(1-3)\boldsymbol{\Sigma}^*(3,4)S_{\mathrm{F}}^{\prime J}(4,2)\,,\end{aligned} \tag{2.4.16}$$

and

$$D_{\mathrm{F}}'^J(1,2) = D_0^{\mathrm{F}}(1-2) + D_0^{\mathrm{F}}(1-3)\boldsymbol{\Pi}^*(3,4)D_{\mathrm{F}}'^J(4,2)\,, \tag{2.4.17}$$

obtaining

$$-G_0\gamma(1)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(1)}S_{\mathrm{F}}'^J(1,2) = \boldsymbol{\Sigma}^*(1,3)S_{\mathrm{F}}'^J(3,2) \equiv \boldsymbol{\Sigma}^*(1)S_{\mathrm{F}}'^J(1,2)\,, \tag{2.4.18}$$

and

$$\begin{aligned} G_0\mathrm{tr}\gamma(1)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(2)}S_{\mathrm{F}}'^J(1,1_\pm) &= \boldsymbol{\Pi}^*(1,3)D_{\mathrm{F}}'^J(3,2) \\ &\equiv \boldsymbol{\Pi}^*(1)D_{\mathrm{F}}'^J(1,2)\,. \end{aligned} \tag{2.4.19}$$

Thus, we can write the functional differential equations, (2.4.12a) and (2.4.14a), compactly as

$$\{D(1) + G_0\gamma(1)\langle\hat{\phi}(1)\rangle^J - \boldsymbol{\Sigma}^*(1)\}S_{\mathrm{F}}'^J(1,2) = \delta(1-2)\,, \tag{2.4.20}$$

and

$$\{K(1) - \boldsymbol{\Pi}^*(1)\}D_{\mathrm{F}}'^J(1,2) = \delta(1-2)\,. \tag{2.4.21}$$

Defining the *nucleon* differential operator and *meson* differential operator by

$$D_N(1,2) \equiv \{D(1) + G_0\gamma(1)\langle\hat{\phi}(1)\rangle^J\}\delta(1-2) - \boldsymbol{\Sigma}^*(1,2)\,, \tag{2.4.22}$$

and

$$D_M(1,2) \equiv K(1)\delta(1-2) - \boldsymbol{\Pi}^*(1,2)\,, \tag{2.4.23}$$

we can write the differential equations, (2.4.20) and (2.4.21), as

$$D_N(1,3)S_{\mathrm{F}}'^J(3,2) = \delta(1-2), \quad \text{or} \quad D_N(1,2) = (S_{\mathrm{F}}'^J(1,2))^{-1}\,, \tag{2.4.24}$$

and

$$D_M(1,3)D_{\mathrm{F}}'^J(3,2) = \delta(1-2), \quad \text{or} \quad D_M(1,2) = (D_{\mathrm{F}}'^J(1,2))^{-1}\,. \tag{2.4.25}$$

Next, we take the functional derivative of (2.4.20).

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}(2.4.20):$$

$$\begin{aligned} &\{D(1) + G_0\gamma(1)\langle\hat{\phi}(1)\rangle^J - \boldsymbol{\Sigma}^*(1)\}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}S_{\mathrm{F}}'^J(1,2) \\ &\quad = \left\{-\mathrm{i}G_0\gamma(1)D_{\mathrm{F}}'^J(1,3) + \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}\boldsymbol{\Sigma}^*(1)\right\}S_{\mathrm{F}}'^J(1,2)\,. \end{aligned} \tag{2.4.26}$$

Solving (2.4.26) for $\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}S_{\mathrm{F}}^{\prime J}(1,2)$, and making use of (2.4.20), (2.4.22) and (2.4.24), we obtain

$$\begin{aligned}
&\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}S_{\mathrm{F}}^{\prime J}(1,2)\\
&\quad= S_{\mathrm{F}}^{\prime J}(1,4)\left\{-\mathrm{i}G_0\gamma(4)D_{\mathrm{F}}^{\prime J}(4,3)+\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}\boldsymbol{\Sigma}^*(4)\right\}S_{\mathrm{F}}^{\prime J}(4,2)\\
&\quad= -\mathrm{i}G_0 S_{\mathrm{F}}^{\prime J}(1,4)\left\{\gamma(4)\delta(4-5)\delta(4-6)\right.\\
&\qquad\left.-\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(6)\rangle^J}\boldsymbol{\Sigma}^*(4,5)\right\}S_{\mathrm{F}}^{\prime J}(5,2)D_{\mathrm{F}}^{\prime J}(6,3)\,.
\end{aligned}\tag{2.4.27}$$

Comparing (2.4.27) with the definition of the vertex operator $\boldsymbol{\Gamma}(4,5;6)$ of Dyson,

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}S_{\mathrm{F}}^{\prime J}(1,2)\equiv -\mathrm{i}G_0 S_{\mathrm{F}}^{\prime J}(1,4)\boldsymbol{\Gamma}(4,5;6)S_{\mathrm{F}}^{\prime J}(5,2)D_{\mathrm{F}}^{\prime J}(6,3)\,,\tag{2.4.28}$$

we obtain

$$\boldsymbol{\Gamma}(1,2;3)=\gamma(1)\delta(1-2)\delta(1-3)-\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}\boldsymbol{\Sigma}^*(1,2)\,,\tag{2.4.29}$$

while we can write the left-hand side of (2.4.28) as

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(3)}S_{\mathrm{F}}^{\prime J}(1,2)=\mathrm{i}D_{\mathrm{F}}^{\prime J}(6,3)\frac{\delta}{\delta\langle\hat{\phi}(6)\rangle^J}S_{\mathrm{F}}^{\prime J}(1,2)\,.\tag{2.4.30}$$

From this, we have

$$\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(6)\rangle^J}S_{\mathrm{F}}^{\prime J}(1,2)=-S_{\mathrm{F}}^{\prime J}(1,4)\boldsymbol{\Gamma}(4,5;6)S_{\mathrm{F}}^{\prime J}(5,2)\,,$$

and we obtain the compact representation of $\boldsymbol{\Gamma}(1,2;3)$,

$$\begin{aligned}
\boldsymbol{\Gamma}(1,2;3)&=\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}(S_{\mathrm{F}}^{\prime J}(1,2))^{-1}=\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}D_N(1,2)\\
&=(2.4.29)\,.
\end{aligned}\tag{2.4.31}$$

Lastly, from (2.4.18) and (2.4.19), which define $\boldsymbol{\Sigma}^*(1,2)$ and $\boldsymbol{\Pi}^*(1,2)$ indirectly and the defining equation of $\boldsymbol{\Gamma}(1,2;3)$, (2.4.28), we have

$$\boldsymbol{\Sigma}^*(1,3)S_{\mathrm{F}}^{\prime J}(3,2)=iG_0^2\gamma(1)S_{\mathrm{F}}^{\prime J}(1,4)\boldsymbol{\Gamma}(4,5;6)S_{\mathrm{F}}^{\prime J}(5,2)D_{\mathrm{F}}^{\prime J}(6,1)\,,\tag{2.4.32}$$

and

$$\boldsymbol{\Pi}^*(1,3)D_{\mathrm{F}}^{\prime J}(3,2) = -\mathrm{i}G_0^2 \mathrm{tr}\gamma(1)S_{\mathrm{F}}^{\prime J}(1,4)\boldsymbol{\Gamma}(4,5;6) \times S_{\mathrm{F}}^{\prime J}(5,1)D_{\mathrm{F}}^{\prime J}(6,2)\,. \qquad (2.4.33)$$

Namely, we obtain

$$\boldsymbol{\Sigma}^*(1,2) = iG_0^2\gamma(1)S_{\mathrm{F}}^{\prime J}(1,3)\boldsymbol{\Gamma}(3,2;4)D_{\mathrm{F}}^{\prime J}(4,1)\,, \qquad (2.4.34)$$

and

$$\boldsymbol{\Pi}^*(1,2) = -\mathrm{i}G_0^2 \mathrm{tr}\gamma(1)S_{\mathrm{F}}^{\prime J}(1,3)\boldsymbol{\Gamma}(3,4;2)S_{\mathrm{F}}^{\prime J}(4,1)\,. \qquad (2.4.35)$$

**Summary of the Schwinger–Dyson Equations**

$$D_N(1,3)S_{\mathrm{F}}^{\prime J}(3,2) = \delta(1-2), \qquad D_M(1,3)D_{\mathrm{F}}^{\prime J}(3,2) = \delta(1-2)\,,$$

$$D_N(1,2) \equiv \{D(1) + G_0\gamma(1)\langle\hat{\phi}(1)\rangle^J\}\delta(1-2) - \boldsymbol{\Sigma}^*(1,2)\,,$$

$$D_M(1,2) \equiv K(1)\delta(1-2) - \boldsymbol{\Pi}^*(1,2)\,,$$

$$K(1)\langle\hat{\phi}(1)\rangle^J - \mathrm{i}G_0 \mathrm{tr}(\gamma(1)S_{\mathrm{F}}^{\prime J}(1,1)) = -J(1)\,,$$

$$\boldsymbol{\Sigma}^*(1,2) \equiv iG_0^2\gamma(1)S_{\mathrm{F}}^{\prime J}(1,3)\boldsymbol{\Gamma}(3,2;4)D_{\mathrm{F}}^{\prime J}(4,1)\,,$$

$$\boldsymbol{\Pi}^*(1,2) \equiv -\mathrm{i}G_0^2 \mathrm{tr}\{\gamma(1)S_{\mathrm{F}}^{\prime J}(1,3)\boldsymbol{\Gamma}(3,4;2)S_{\mathrm{F}}^{\prime J}(4,1)\}\,,$$

$$\begin{aligned}\boldsymbol{\Gamma}(1,2;3) &\equiv \frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}(S_{\mathrm{F}}^{\prime J}(1,2))^{-1} \\ &= \gamma(1)\delta(1-2)\delta(1-3) - \frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}\boldsymbol{\Sigma}^*(1,2)\,.\end{aligned}$$

This system of nonlinear coupled integro-differential equations is exact and closed. Starting from the $0^{\text{th}}$ order term of $\boldsymbol{\Gamma}(1,2;3)$, we can develop covariant perturbation theory by iterations.

## 2.5 Equivalence of Path Integral Quantization and Canonical Quantization

In this section, starting with the path integral quantization of Feynman as *Feynman's action principle*, we derive the canonical quantization from the path integral quantization for the nonsingular Lagrangian (density). From the discussions of Sects. 2.1 and 2.3, we shall see that path integral quantization and canonical quantization are equivalent, at least for the nonsingular Lagrangian (density).

In Sect. 2.5.1, we shall state Feynman's action principle and we make three assumptions which will be acceptable as common sense:

(A.1) principle of superposition and the composition law;
(A.2) functional integration by parts;
(A.3) resolution of the identity.

In Sect. 2.5.2, we derive the definition of the field operator in terms of its matrix elements, the Euler–Lagrange equation of motion, and the definition of the time-ordered product in terms of its matrix elements from path integral quantization.

In Sect. 2.5.3, we show that the definition of the momentum operator as the displacement operator in path integral quantization agrees with the definition of the canonically conjugate momentum in canonical quantization, and we derive the equal-time canonical (anti-)commutators from path integral quantization.

In Sects. 2.1 and 2.3, by two different approaches, we have shown that

$$\text{canonical quantization} \quad \Rightarrow \quad \text{path integral quantization},$$

while in this section, we shall show that

$$\text{canonical quantization} \quad \Leftarrow \quad \text{path integral quantization}.$$

Thus, we shall demonstrate their equivalence at least for a nonsingular Lagrangian (density).

In this section, in order to emphasize the parallelism of the discussions of Feynman's action principle in quantum mechanics and quantum field theory, we shall add "M" and "F" at the end of each equation number.

### 2.5.1 Feynman's Action Principle

The operator $\hat{q}(t)$ for all time $t$ (the operator $\hat{\phi}(x)$ for all space-time indices $x$ on a space-like hypersurface $\sigma$) forms a complete set of commuting operators. In other words, the quantum-theoretical state vector can be expressed in terms of the complete set of eigenkets $|q,t\rangle$ of the commuting operators $\hat{q}(t)$ (the complete set of eigenkets $|\phi,\sigma\rangle$ of the commuting operators $\hat{\phi}(x)$). Feynman's action principle declares that the transformation function $\langle q'',t''|q',t'\rangle$

($\langle \phi'', \sigma'' | \phi', \sigma' \rangle$) is given by

$$\langle q'', t'' | q', t' \rangle = \frac{1}{N(\Omega)} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q(t)] \times \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \mathrm{d}t L\left( q(t), \frac{\mathrm{d}}{\mathrm{d}t} q(t) \right) \right], \tag{2.5.1M}$$

$$\langle \phi'', \sigma'' | \phi', \sigma' \rangle = \frac{1}{N(\Omega)} \int_{\phi', \sigma'}^{\phi'', \sigma''} \mathcal{D}[\phi] \times \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{\Omega(\sigma'', \sigma')} \mathrm{d}^4 x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \right], \tag{2.5.1F}$$

where $\Omega$ is given by

$$\Omega = \begin{cases} \Omega(t'', t') = \text{ space-time region sandwiched between } t' \text{ and } t'', \\ \Omega(\sigma'', \sigma') = \text{ space-time region sandwiched between } \sigma'' \text{ and } \sigma'. \end{cases}$$

We state three assumptions here.

(A-1) The principle of superposition and the composition law of the probability amplitudes in the space-time region $\Omega_{\mathrm{I}} + \Omega_{\mathrm{II}}$, where $\Omega_{\mathrm{I}}$ and $\Omega_{\mathrm{II}}$ are neighbors:

$$\int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q] = \int_{t=t'''} \mathrm{d}q''' \int_{q_{\mathrm{II}}(t''')=q'''}^{q_{\mathrm{II}}(t'')=q''} \times \mathcal{D}[q_{\mathrm{II}}] \int_{q_{\mathrm{I}}(t')=q'}^{q_{\mathrm{I}}(t''')=q'''} \mathcal{D}[q_{\mathrm{I}}], \tag{2.5.2M}$$

$$\int_{\phi', \sigma'}^{\phi'', \sigma''} \mathcal{D}[\phi] = \int_{\sigma'''} \mathrm{d}\phi''' \int_{\phi''', \sigma'''}^{\phi'', \sigma''} \mathcal{D}[\phi_{\mathrm{II}}] \int_{\phi', \sigma'}^{\phi''', \sigma'''} \mathcal{D}[\phi_{\mathrm{I}}]. \tag{2.5.2F}$$

(A-2) Functional integration by parts is allowed. The requisite "i$\varepsilon$-piece" which damps the contributions from the functional infinities is already included in the Lagrangian (density).

(A-3) Resolution of the identity:

$$\int \mathrm{d}q' | q', t' \rangle \langle q', t' | = 1, \tag{2.5.3M}$$

$$\int \mathrm{d}\phi' | \phi', \sigma' \rangle \langle \phi', \sigma' | = 1. \tag{2.5.3F}$$

From the consistency of these three assumptions, the normalization constant $N(\Omega)$ must satisfy

$$N(\Omega_1 + \Omega_2) = N(\Omega_1)N(\Omega_2)\,, \tag{2.5.4}$$

which also originates from the additivity of the action functional,

$$I[q; t'', t'] = I[q; t'', t'''] + I[q; t''', t']\,, \tag{2.5.5M}$$

$$I[\phi; \sigma'', \sigma'] = I[\phi; \sigma'', \sigma'''] + I[\phi; \sigma''', \sigma']\,. \tag{2.5.5F}$$

The action functional is defined by

$$I[q; t'', t'] = \int_{t'}^{t''} \mathrm{d}t L\left(q(t), \frac{\mathrm{d}}{\mathrm{d}t}q(t)\right)\,, \tag{2.5.6M}$$

$$I[\phi; \sigma'', \sigma'] = \int_{\Omega(\sigma'', \sigma')} \mathrm{d}^4 x \mathcal{L}(\phi(x), \partial_\mu \phi(x))\,. \tag{2.5.6F}$$

### 2.5.2 The Operator, Equation of Motion and Time-Ordered Product

In path integral quantization, the operator is defined by its matrix elements:

$$\begin{aligned}\langle q'', t''|\hat{q}(t)|q', t'\rangle = &\frac{1}{N(\Omega)} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q] q(t) \\ &\times \exp\left[\frac{\mathrm{i}}{\hbar} I[q; t'', t']\right]\,,\end{aligned} \tag{2.5.7M}$$

$$\begin{aligned}\langle \phi'', \sigma''|\hat{\phi}(x)|\phi', \sigma'\rangle = &\frac{1}{N(\Omega)} \int_{\phi', \sigma'}^{\phi'', \sigma''} \mathcal{D}[\phi] \phi(x) \\ &\times \exp\left[\frac{\mathrm{i}}{\hbar} I[\phi; \sigma'', \sigma']\right]\,.\end{aligned} \tag{2.5.7F}$$

If $t$ lies on the $t''$-surface ($x$ lies on the $\sigma''$-surface), (2.5.7M) and (2.5.7F) can be rewritten on the basis of (2.5.1M) and (2.5.1F) as

$$\langle q'', t''|\hat{q}(t'')|q', t'\rangle = q''\langle q'', t''|q', t'\rangle\,, \tag{2.5.8M}$$

$$\langle \phi'', \sigma''|\hat{\phi}(x'')|\phi', \sigma'\rangle = \phi''(x'')\langle \phi'', \sigma''|\phi', \sigma'\rangle\,, \tag{2.5.8F}$$

for all $|q', t'\rangle$ (for all $|\phi', \sigma'\rangle$) which form a complete set. Then, we have

$$\langle q'', t''|\hat{q}(t'') = q''\langle q'', t''| , \tag{2.5.9M}$$

$$\langle \phi'', \sigma''|\hat{\phi}(x'') = \phi''(x'')\langle \phi'', \sigma''| , \tag{2.5.9F}$$

which are the defining equations of the eigenbras, $\langle q'', t''|$ and $\langle \phi'', \sigma''|$.

Next, we consider the variation of the action functional.

$$\begin{aligned} \delta I[q; t'', t'] =& \int_{t'}^{t''} \mathrm{d}t \left\{ \frac{\partial L(q(t), \mathrm{d}q(t)/\mathrm{d}t)}{\partial q(t)} - \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial L\left(q(t), \mathrm{d}q(t)/\mathrm{d}t\right)}{\partial\left(\mathrm{d}q(t)/\mathrm{d}t\right)} \right) \right\} \delta q(t) \\ &+ \int_{t'}^{t''} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \frac{\partial L\left(q(t), \mathrm{d}q(t)/\mathrm{d}t\right)}{\partial\left(\mathrm{d}q(t)/\mathrm{d}t\right)} \delta q(t) \right\} , \end{aligned} \tag{2.5.10M}$$

$$\begin{aligned} \delta I[\phi; \sigma'', \sigma'] = & \int_{\Omega(\sigma'', \sigma')} \mathrm{d}^4x \left\{ \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} \right. \\ & \left. -\partial_\mu \left( \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))} \right) \right\} \delta\phi(x) \\ & + \int_{\Omega(\sigma'', \sigma')} \mathrm{d}^4x \partial_\mu \left\{ \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))} \delta\phi(x) \right\} . \end{aligned} \tag{2.5.10F}$$

We consider particular variations in which the end-points are fixed,

$$\delta q(t') = \delta q(t'') = 0; \quad \delta\phi(x' \text{ on } \sigma') = \delta\phi(x'' \text{ on } \sigma'') = 0 . \tag{2.5.11}$$

Then, the second terms in (2.5.10M) and (2.5.10F) vanish. We obtain the Euler derivatives,

$$\frac{\delta I[q; t'', t']}{\delta q(t)} = \frac{\partial L\left(q(t), \mathrm{d}q(t)/\mathrm{d}t\right)}{\partial q(t)} - \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial L\left(q(t), \mathrm{d}q(t)/\mathrm{d}t\right)}{\partial\left(\frac{\mathrm{d}}{\mathrm{d}t} q(t)\right)} \right) ,$$

$$\frac{\delta I[\phi; \sigma'', \sigma']}{\delta\phi(x)} = \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} - \partial_\mu \left( \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial\left(\partial_\mu \phi(x)\right)} \right) .$$

Now, we evaluate the matrix elements of the operators,

$$\frac{\delta I[\hat{q}; t'', t']}{\delta \hat{q}(t)} \quad \text{and} \quad \frac{\delta I[\hat{\phi}; \sigma'', \sigma']}{\delta \hat{\phi}(x)} ,$$

in accordance with (2.5.7M) and (2.5.7F).

$$\begin{aligned}&\left\langle q'',t''\left|\frac{\delta I[\hat{q};t'',t']}{\delta\hat{q}(t)}\right|q',t'\right\rangle\\&\quad=\frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\frac{\delta I[q;t'',t']}{\delta q(t)}\exp\left[\frac{\mathrm{i}}{\hbar}I[q;t'',t']\right]\\&\quad=\frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta q(t)}\exp\left[\frac{\mathrm{i}}{\hbar}I[q;t'',t']\right]\\&\quad=0\,,\end{aligned}\tag{2.5.12M}$$

$$\begin{aligned}&\left\langle \phi'',\sigma''\left|\frac{\delta I[\hat{\phi};\sigma'',\sigma']}{\delta\hat{\phi}(x)}\right|\phi',\sigma'\right\rangle\\&\quad=\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}[\phi]\frac{\delta I[\phi;\sigma'',\sigma']}{\delta \phi(x)}\exp\left[\frac{\mathrm{i}}{\hbar}I[\phi;\sigma'',\sigma']\right]\\&\quad=\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}[\phi]\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta \phi(x)}\exp\left[\frac{\mathrm{i}}{\hbar}I[\phi;\sigma'',\sigma']\right]\\&\quad=0\,.\end{aligned}\tag{2.5.12F}$$

By the assumption that the eigenkets $|q,t\rangle$ ($|\phi,\sigma\rangle$) form a complete set of the expansion basis, we obtain the *Euler–Lagrange equations of motion at the operator level*, from (2.5.12M) and (2.5.12F), as

$$\frac{\delta I[\hat{q}]}{\delta\hat{q}(t)}=0,\qquad\text{and}\qquad\frac{\delta I[\hat{\phi}]}{\delta\hat{\phi}(x)}=0\,.\tag{2.5.13}$$

As for the time-ordered products, we have

$$\begin{aligned}&\frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]q(t_1)\cdots q(t_n)\exp\left[\frac{\mathrm{i}}{\hbar}I[q;t'',t']\right]\\&\quad=\langle q'',t''|\mathrm{T}(\hat{q}(t_1)\cdots\hat{q}(t_n))|q',t'\rangle\,,\end{aligned}\tag{2.5.14M}$$

$$\begin{aligned}&\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}[\phi]\phi(x_1)\cdots \phi(x_n)\exp\left[\frac{\mathrm{i}}{\hbar}I[\phi;\sigma'',\sigma']\right]\\&\quad=\langle \phi'',\sigma''|\mathrm{T}(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))|\phi',\sigma'\rangle\,.\end{aligned}\tag{2.5.14F}$$

We can prove above by mathematical induction starting from $n=2$. Again as in Sect. 1.2.3, the objects given by the left-hand sides of Eqs. (2.5.14M) and (2.5.14F) are the matrix elements of the canonical $\mathrm{T}^*$-product which is defined for quantum field theory by the following equations,

$$\mathrm{T}^*(\partial_\mu^{x_1}\hat{O}_1(x_1)\cdots\hat{O}_n(x_n)) \equiv \partial_\mu^{x_1}\mathrm{T}^*(\hat{O}_1(x_1)\cdots\hat{O}_n(x_n))\,,$$

$$\mathrm{T}^*(\hat{\phi}_{r_1}(x_1)\cdots\hat{\phi}_{r_n}(x_n)) \equiv \mathrm{T}(\hat{\phi}_{r_1}(x_1)\cdots\hat{\phi}_{r_n}(x_n))\,,$$

with

$$\mathrm{T}(\hat{\phi}_{r_1}(x)\hat{\phi}_{r_2}(y)) \equiv \theta(x^0-y^0)\hat{\phi}_{r_1}(x)\hat{\phi}_{r_2}(y)+\theta(y^0-x^0)\hat{\phi}_{r_2}(y)\hat{\phi}_{r_1}(x)\,.$$

### 2.5.3 Canonically Conjugate Momentum and Equal-Time Canonical (Anti-)Commutators

We define the momentum operator as the displacement operator,

$$\langle q'',t''|\hat{p}(t'')|q',t'\rangle = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q''}\langle q'',t''|q',t'\rangle\,, \tag{2.5.15M}$$

$$\langle \phi'',\sigma''|\hat{\pi}(x'')|\phi',\sigma'\rangle = \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta\phi''}\langle \phi'',\sigma''|\phi',\sigma'\rangle\,. \tag{2.5.15F}$$

In order to express the right-hand sides of (2.5.15M) and (2.5.15F) in terms of the path integral representation, we consider the following variation of the action functional.

(1) Inside $\Omega$, we consider infinitesimal variations of $q(t)$ and $\phi(x)$,

$$q(t)\to q(t)+\delta q(t), \qquad \text{and} \qquad \phi(x)\to\phi(x)+\delta\phi(x)\,, \tag{2.5.16}$$

where, as $\delta q(t)$ and $\delta\phi(x)$, we take

$$\delta q(t')=0, \qquad \delta q(t)=\xi(t), \qquad \delta q(t'')=\xi''\,, \tag{2.5.17M}$$

$$\delta\phi(x')=0, \qquad \delta\phi(x)=\xi(x), \qquad \delta\phi(x'')=\xi''\,. \tag{2.5.17F}$$

(2) Inside $\Omega$, the physical system evolves in time in accordance with the *Euler–Lagrange equation of motion.*

As the response of the action functional to particular variations, (1) and (2), we have

$$\delta I[q;t'',t'] = \frac{\partial L(q(t''),\mathrm{d}q(t'')/\mathrm{d}t'')}{\partial(\mathrm{d}q(t'')/\mathrm{d}t'')}\xi''\,, \tag{2.5.18M}$$

$$\delta I[\phi;\sigma'',\sigma'] = \int_{\sigma''}\mathrm{d}\sigma''_\mu\left\{\frac{\partial\mathcal{L}(\phi(x''),\partial_\mu\phi(x''))}{\partial(\partial_\mu\phi(x''))}\xi''\right\}\,, \tag{2.5.18F}$$

where the first terms of (2.5.10M) and (2.5.10F) vanish due to the Euler–Lagrange equation of motion.

Thus, we obtain

$$\frac{\delta I[q;t'',t']}{\delta q(t'')} = \frac{\partial L(q(t''),\mathrm{d}q(t'')/\mathrm{d}t'')}{\partial(\mathrm{d}q(t'')/\mathrm{d}t'')}\,, \tag{2.5.19M}$$

$$\frac{\delta I[\phi;\sigma'',\sigma']}{\delta\phi(x'')} = n_\mu(x'')\frac{\partial\mathcal{L}(\phi(x''),\partial_\mu\phi(x''))}{\partial(\partial_\mu\phi(x''))}\,, \tag{2.5.19F}$$

where $n_\mu(x'')$ is the unit normal vector at a point $x''$ on the space-like surface $\sigma''$. With these preparations, we can express the right-hand sides of (2.5.15M) and (2.5.15F) as

$$\begin{aligned}
&\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q(t'')}\langle q'',t''|q',t'\rangle \\
&\quad= \frac{\hbar}{\mathrm{i}}\lim_{\xi''\to 0}\frac{\langle q''+\xi'',t''|q',t'\rangle - \langle q'',t''|q',t'\rangle}{\xi''} \\
&\quad= \frac{\hbar}{\mathrm{i}}\lim_{\xi''\to 0}\frac{1}{N(\varOmega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\exp\left[\frac{\mathrm{i}}{\hbar}I[q;t'',t']\right] \\
&\qquad\times\frac{1}{\xi''}\left\{\exp\left[\frac{\mathrm{i}}{\hbar}(I[q+\xi;t'',t'] - I[q;t'',t'])\right]-1\right\} \\
&\quad= \lim_{\xi''\to 0}\frac{1}{N(\varOmega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\exp\left[\frac{\mathrm{i}}{\hbar}I[q;t'',t']\right] \\
&\qquad\times\frac{I[q+\xi;t'',t'] - I[q;t'',t'] + O((\xi'')^2)}{\xi''} \\
&\quad= \frac{1}{N(\varOmega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\exp\left[\frac{\mathrm{i}}{\hbar}I[q;t'',t']\right]\frac{\delta I[q;t'',t']}{\delta q(t'')} \\
&\quad= \frac{1}{N(\varOmega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\exp\left[\frac{\mathrm{i}}{\hbar}I[q;t'',t']\right] \\
&\qquad\times\frac{\partial L(q(t''),\,\mathrm{d}q(t'')/\mathrm{d}t''))}{\partial(\mathrm{d}q(t'')/\mathrm{d}t'')} \\
&\quad= \left\langle q'',t''\left|\frac{\partial L(\hat{q}(t''),\mathrm{d}\hat{q}(t'')/\mathrm{d}t''))}{\partial(\mathrm{d}\hat{q}(t'')/\mathrm{d}t''))}\right|q',t'\right\rangle\,.
\end{aligned} \tag{2.5.20M}$$

Likewise, we obtain

$$\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta\phi''(x'')}\langle\phi'',\sigma''|\phi',\sigma'\rangle$$
$$=\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}[\phi]\exp\left[\frac{\mathrm{i}}{\hbar}I[\phi;\sigma'',\sigma']\right]\frac{\delta I[\phi;\sigma'',\sigma']}{\delta\phi(x'')}$$
$$=\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}[\phi]\exp\left[\frac{\mathrm{i}}{\hbar}I[\phi;\sigma'',\sigma']\right]n_\mu(x'')$$
$$\times\frac{\partial\mathcal{L}(\phi(x''),\partial_\mu\phi(x''))}{\partial(\partial_\mu\phi(x''))}$$
$$=\left\langle\phi'',\sigma''\left|n_\mu(x'')\frac{\partial\mathcal{L}(\hat{\phi}(x''),\partial_\mu\hat{\phi}(x''))}{\partial(\partial_\mu\hat{\phi}(x''))}\right|\phi',\sigma'\right\rangle. \tag{2.5.20F}$$

From (2.5.15M) and (2.5.15F), and (2.5.20M) and (2.5.20F), we obtain the operator identities,

$$\hat{p}(t)=\frac{\partial L(\hat{q}(t),\,\mathrm{d}\hat{q}(t)/\mathrm{d}t)}{\partial(\mathrm{d}\hat{q}(t)/\mathrm{d}t)}, \tag{2.5.21M}$$

$$\hat{\pi}(x)=n_\mu(x)\frac{\partial\mathcal{L}(\hat{\phi}(x),\partial_\mu\hat{\phi}(x))}{\partial(\partial_\mu\hat{\phi}(x))}. \tag{2.5.21F}$$

Equation (2.5.21M) is the definition of the momentum $\hat{p}(t)$ canonically conjugate to $\hat{q}(t)$. With the choice of the unit normal vector as

$$n_\mu(x)=(1,0,0,0),$$

(2.5.21F) is also the definition of the momentum $\hat{\pi}(x)$ canonically conjugate to $\hat{\phi}(x)$. A noteworthy point is that $\hat{\pi}(x)$ is a *normal dependent quantity.*

Lastly, we discuss the equal-time canonical (anti-)commutators. For the quantum mechanics of a *Bose particle* system, from

$$\langle q'',t''|\hat{q}_{\mathrm{B}}(t'')|q',t'\rangle=q''_{\mathrm{B}}\langle q'',t''|q',t'\rangle \tag{2.5.22M}$$

and from (2.5.15M), we have

$$\langle q'',t''|\hat{p}_{\mathrm{B}}(t'')\hat{q}_{\mathrm{B}}(t'')|q',t'\rangle$$
$$=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q''_{\mathrm{B}}}(q''_{\mathrm{B}}\langle q'',t''|q',t'\rangle)$$
$$=\frac{\hbar}{\mathrm{i}}\langle q'',t''|q',t'\rangle+q''_{\mathrm{B}}\langle q'',t''|\hat{p}_{\mathrm{B}}(t'')|q',t'\rangle$$
$$=\frac{\hbar}{\mathrm{i}}\langle q'',t''|q',t'\rangle+\langle q'',t''|\hat{q}_{\mathrm{B}}(t'')\hat{p}_{\mathrm{B}}(t'')|q',t'\rangle, \tag{2.5.23M}$$

i.e., we have

$$[\hat{p}_{\mathrm{B}}(t), \hat{q}_{\mathrm{B}}(t)] = \frac{\hbar}{\mathrm{i}} . \tag{2.5.24M}$$

For the quantum mechanics of a *Fermi particle* system, we have extra minus signs on the second terms of the second and third lines of (2.5.23M), arising from the anticommutativity of the Fermion number, and thus we obtain

$$\{\hat{p}_{\mathrm{F}}(t), \hat{q}_{\mathrm{F}}(t)\} = \frac{\hbar}{\mathrm{i}} . \tag{2.5.25M}$$

For quantum field theory of the *Bose field*, from

$$\langle \phi'', \sigma'' | \hat{\phi}_{\mathrm{B}}(x'') | \phi', \sigma' \rangle = \phi''_{\mathrm{B}}(x'') \langle \phi'', \sigma'' | \phi', \sigma' \rangle \tag{2.5.22F}$$

and from (2.5.15F), we have

$$\begin{aligned} &\langle \phi'', \sigma'' | \hat{\pi}_{\mathrm{B}}(x''_1) \hat{\phi}_{\mathrm{B}}(x''_2) | \phi', \sigma' \rangle \\ &= \frac{\hbar}{\mathrm{i}} \frac{\delta}{\delta \phi''_{\mathrm{B}}(x''_1)} \left( \phi''_{\mathrm{B}}(x''_2) \langle \phi'', \sigma'' | \phi', \sigma' \rangle \right) \\ &= \frac{\hbar}{\mathrm{i}} \delta_{\sigma''}(x''_1 - x''_2) \langle \phi'', \sigma'' | \phi', \sigma' \rangle + \phi''_{\mathrm{B}}(x''_2) \langle \phi'', \sigma'' | \hat{\pi}_{\mathrm{B}}(x''_1) | \phi', \sigma' \rangle \\ &= \frac{\hbar}{\mathrm{i}} \delta_{\sigma''}(x''_1 - x''_2) \langle \phi'', \sigma'' | \phi', \sigma' \rangle + \langle \phi'', \sigma'' | \hat{\phi}_{\mathrm{B}}(x''_2) \hat{\pi}_{\mathrm{B}}(x''_1) | \phi', \sigma' \rangle , \end{aligned} \tag{2.5.23F}$$

i.e., we have

$$[\hat{\pi}_{\mathrm{B}}(x_1), \hat{\phi}_{\mathrm{B}}(x_2)] = \frac{\hbar}{\mathrm{i}} \delta_{\sigma}(x_1 - x_2) . \tag{2.5.24F}$$

For quantum field theory of the *Fermi field*, for the same reason as in (2.5.25M), we obtain

$$\{\hat{\pi}_{\mathrm{F}}(x_1), \hat{\phi}_{\mathrm{F}}(x_2)\} = \frac{\hbar}{\mathrm{i}} \delta_{\sigma}(x_1 - x_2) . \tag{2.5.25F}$$

In summary, from Feynman's action principle, (2.5.1M,F), the assumptions (A-1), (A-2) and (A-3), the definition of the operator in the path integral quantization of Feynman, (2.5.7M,F), and the definition of the momentum operator as the displacement operator, (2.5.15M,F), we deduced

(a) the Euler–Lagrange equation of motion, (2.5.13),
(b) the definition of the time-ordered product, (2.5.14M,F),
(c) the definition of the canonical conjugate momentum, (2.5.21M,F), and
(d) the equal-time canonical (anti-)commutator, (2.5.24M,F) and (2.5.25M,F).

Thus, we have demonstrated

$$\text{canonical quantization} \quad \Leftarrow \quad \text{path integral quantization.}$$

In view of the discussions in Sects. 2.1, 2.3 and in this section, we have shown the equivalence of path integral quantization and canonical quantization for the nonsingular Lagrangian (density).

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# 3. Path Integral Quantization of Gauge Field

In Chap. 2, we assumed that the Lagrangian density

$$\mathcal{L}(\psi(x), \partial_\mu \psi(x))$$

is nonsingular. In this chapter, we discuss the path integral quantization of gauge field theory. Gauge field theory is singular in the sense that the kernel of the quadratic part of the gauge field Lagrangian density is a four-dimensionally transverse projection operator, which is not invertible without the gauge fixing term.

The most familiar example of a gauge field is the electrodynamics of J.C. Maxwell. The electromagnetic field has a well-known invariance property under the gauge transformation (gauge invariance). The charged matter field interacting with the electromagnetic field has a property known as the charge conservation law. The original purpose of the introduction of the notion of gauge invariance (originally called Eichinvarianz) by H. Weyl was a unified description of the gauge invariance of the electromagnetic field and the scale invariance of the gravitational field. Weyl failed to accomplish his goal. After the birth of quantum mechanics, however, Weyl reconsidered gauge invariance and discovered that the gauge invariance of the electromagnetic field is not related to the scale invariance of the gravitational field, but is related to the local phase transformation of the matter field and that the interacting system of the electromagnetic field and the matter field is invariant under the gauge transformation of the electromagnetic field plus the local phase transformation of the matter field. The invariance of the matter field Lagrangian density under a global phase transformation results in the charge conservation law, according to Noether's theorem. Weyl's gauge principle declares that the extension of the global invariance (invariance under the space-time independent phase transformation) of the matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu \psi(x))$$

to the local invariance (invariance under the space-time dependent phase transformation) of the matter field Lagrangian density under the continuous symmetry group $G$ necessitates the introduction of the gauge field and the replacement of the derivative $\partial_\mu \psi(x)$ with the covariant derivative $D_\mu \psi(x)$ in the matter field Lagrangian density,

M. Masujima: Path Integral Quantization and Stochastic Quantization,
STMP 165, 103–174 (2000)

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu \psi(x)) \rightarrow \mathcal{L}_{\text{matter}}(\psi(x), D_\mu \psi(x)).$$

It also requires that the covariant derivative $D_\mu \psi(x)$ transforms exactly like the matter field $\psi(x)$ under the local phase transformation. From this, the transformation law of the gauge field is uniquely determined. It further declares that the coupling of the matter field with the gauge field is universal and that the mechanical mass term of the gauge field is forbidden. As the continuous symmetry group $G$, Weyl restricted his consideration to the $U(1)$ gauge group, the electrodynamics. C.N. Yang and R.L. Mills considered the $SU(2)$-isospin gauge group, and their gauge field is called the Yang–Mills gauge field. Generally, $G$ is a semisimple Lie group. The gauge field transforms under the adjoint representation of $G$, and the matter field undergoes the phase transformation under its representation of $G$.

In this chapter, since the semisimple Lie group plays an important role, we briefly review group theory in Sect. 3.1. We discuss group theory in general (Sect. 3.1.1). We discuss sufficient details of Lie groups (Sect. 3.1.2). In Sect. 3.2, we discuss non-Abelian gauge field theory. We discuss the motivation for the extension of $U(1)$ gauge field theory to $SU(2)$-isospin gauge field theory (Sect. 3.2.1). We discuss the construction of the general non-Abelian gauge field theory with Weyl's gauge principle (Sect. 3.2.2). We compare Abelian gauge field theory with non-Abelian gauge field theory (Sect. 3.2.3). We present explicit examples of $SU(2)$-isospin gauge field theory and $SU(3)$-color gauge field theory (Sect. 3.2.4).

In Sect. 3.3, we discuss path integral quantization of gauge field theory. We cannot accomplish proper quantization of gauge field theory by the application of naive Feynman path integral quantization. In non-Abelian gauge field theory, we obtain nonunitary results, i.e., the conservation of the probability is violated. This difficulty has its origin in gauge invariance. When we perform path integration along the gauge equivalent class (which we call the orbit of the gauge field), the integrand of the functional integral for the vacuum-to-vacuum transition amplitude

$$\langle 0, \text{out} | 0, \text{in} \rangle$$

remains constant and the infinite orbit volume $V_G$ is calculated. Thus, we choose the hypersurface (the gauge fixing condition) which intersects with each orbit only once in the manifold of the gauge field, and we perform the path integration and the group integration on this hypersurface, extracting the orbit volume $V_G$, and completing path integral quantization (Sect. 3.3.1). This method of path integral quantization is known as the Faddeev–Popov method, and results in the emergence of the Faddeev–Popov determinant factor in the functional integrand. This determinant factor can be exponentiated with the introduction of the Faddeev–Popov ghost field, and can be regarded as the addition of the Faddeev–Popov ghost Lagrangian density

$$\mathcal{L}_{\text{FP ghost}}$$

to

$$\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x)) + \mathcal{L}_{\text{gauge}},$$

thus restoring the unitarity. We note that the Faddeev–Popov ghost shows up only in the internal loop. Next, we generalize the gauge fixing condition and show that the vacuum-to-vacuum transition amplitude

$$\langle 0, \text{out} | 0, \text{in} \rangle_{F,a}$$

is invariant under an infinitesimal change of the generalized gauge fixing condition. From this invariance, we derive the second Faddeev–Popov formula (Sect. 3.3.2). We can introduce the quasi-Gaussian functional of the gauge-fixing function and we obtain the total effective Lagrangian density

$$\mathcal{L}_{\text{eff}} \equiv \mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x)) + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{gauge fixing}} + \mathcal{L}_{\text{F.P. ghost}}.$$

Among many gauge-fixing conditions, we discuss the axial gauge, the Landau gauge and the covariant gauge (Sect. 3.3.3). We then discuss the gauge independence of the physical $S$-matrix with the use of the Ward–Takahashi–Slavnov–Taylor identity (Sect. 3.3.4).

In our discussion so far, we have assumed the exact $G$ invariance of the vacuum $|0,{}^{\text{in}}_{\text{out}}\rangle$, and hence, the gauge field remains massless even at the quantum level. In Sect. 3.4, we discuss spontaneous symmetry breaking and gauge field theory, thereby generating a mass term to the gauge field without violating gauge invariance. When the global $G$ invariance of the matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))$$

is spontaneously broken by the vacuum $|0,{}^{\text{in}}_{\text{out}}\rangle$, namely, when $\psi(x)$ develops a nonzero vacuum expectation value with respect to some internal degrees of freedom of $G$, we have the Nambu–Goldstone boson (massless excitation) for each broken generator of $G$ (Sect. 3.4.1). We call this Goldstone's theorem. The elimination of the Nambu–Goldstone boson becomes the main issue. Here, the symmetry is exact, but spontaneously broken. With the use of Weyl's gauge principle, we extend the global $G$ invariance of the matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))$$

to local $G$ invariance of the matter field Lagrangian density, with the introduction of a gauge field and the replacement of the derivative $\partial_\mu\psi(x)$ by the covariant derivative $D_\mu\psi(x)$. When we perform the gauge transformation plus the local phase transformation which depend appropriately on the vacuum expectation value of $\psi(x)$, we can absorb the Nambu–Goldstone boson into the longitudinal degree of freedom of the gauge field corresponding to the broken generator. The said gauge field becomes a massive vector field with three degrees of freedom consisting of two transverse modes and one longitudinal mode. The gauge field corresponding to the unbroken generator remains as a massless gauge field with two degrees of freedom of the two transverse modes. There are no changes in the total number of degrees of freedom in the matter-gauge system since the Nambu–Goldstone boson gets eliminated and emerges as the longitudinal mode of the massive vector field. We call this method of the mass generating mechanism the Higgs–Kibble mechanism (Sect. 3.4.2), which forms one of the foundations of the electroweak unification and the grand unified theories.

Lastly, we discuss path integral quantization of the gauge field in the presence of spontaneous symmetry breaking in the $R_\xi$-gauge (Sect. 3.4.3). The $R_\xi$-gauge is a convenient gauge which fixes the gauge as well as eliminating the mixed term of the gauge field and the matter field in the quadratic part of the total effective Lagrangian density, after invoking the Higgs–Kibble mechanism. The $R_\xi$-gauge depends on the nonnegative real parameter $\xi$. The gauge independence (or $\xi$-independence) of the physical $S$-matrix can be proven with the use of the Ward–Takahashi–Slavnov–Taylor identity (Sect. 3.4.4).

For the following three topics, we refer the reader to S. Weinberg, *The Quantum Theory of Fields*, Vol. II:

(1) Algebraic proof of the renormalizability of the most general renormalizable Lagrangian density, with the use of the Zinn–Justin equation and BRST invariance.
(2) Demonstration of the asymptotic freedom of the non-Abelian gauge field theory, with the use of a background field gauge.
(3) Triangular anomaly of the axial vector current, with the use of the consistency condition.

Path integral quantization of the gauge field with the Faddeev–Popov method has an interesting analog in multivariate normal analysis of mathematical statistics. In multivariate normal analysis, we project the $m$-dimensional “singular” multivariate normal distribution, $\mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, down to a lower $n$-dimensional subspace, and obtain an $n$-dimensional nonsingular multivariate normal distribution, $\mathcal{N}_n(\boldsymbol{\mu}', \boldsymbol{\Sigma}')$. The $(m-n)$-zero eigenvalues of $\boldsymbol{\Sigma}$ get eliminated in this projection. Path integral quantization of the gauge field with the Faddeev–Popov method corresponds to the quasi-Gaussian functional version of this projection. The Faddeev–Popov determinant corresponds to the Jacobian of the coordinate transformation of this projection. On this matter, we refer the reader to Appendices 1 and 6.

## 3.1 Review of Lie Groups

Semisimple Lie groups play a major role in non-Abelian gauge field theory. We discuss group theory in general in Sect. 3.1.1. Starting with a compact Lie group, we discuss Lie groups in particular in Sect. 3.1.2.

### 3.1.1 Group Theory

The group $G$ is the set whose elements satisfy the following four properties with the group multiplication law $\circledcirc$:

(1) $\forall g_1, g_2 \in G : g_1 \circledcirc g_2 \in G$,
(2) $\exists e \in G : e \circledcirc g = g \circledcirc e = g,\ \forall g \in G$,
(3) $\forall g \in G, \exists g^{-1} \in G : g \circledcirc g^{-1} = g^{-1} \circledcirc g = e$,
(4) $\forall g_1, g_2, g_3 \in G : (g_1 \circledcirc g_2) \circledcirc g_3 = g_1 \circledcirc (g_2 \circledcirc g_3)$.

In (1), if we have commutativity,

(1′) $\forall g_1, g_2 \in G : g_1 \circledcirc g_2 = g_2 \circledcirc g_1$,

$G$ is called an Abelian group, otherwise a non-Abelian group.

In quantum theory, there exists a mapping from the group $G$ to the set of abstract linear unitary operators on a Hilbert space, i.e.,

(5) $\forall g \in G : \exists U(g)$, an abstract linear unitary operator on a Hilbert space.

The group multiplication law of the elements of $G$ is preserved under this mapping:

(1″) $\forall g_1, g_2 \in G : U(g_1)U(g_2) = U(g_1 \circledcirc g_2)$.

We call this abstract linear unitary operator $U(g)$ the representation of $G$ on the Hilbert space.

Aside from this representation $U(g)$ on the Hilbert space, there exists a matrix representation $D(g)$. When we choose $\{|n\rangle\}$ as the orthonormal basis of the Hilbert space on which the abstract linear unitary operator $U(g)$ acts, we define the $(n, m)$ element of $D(g)$ by

$$D(g)_{n,m} \equiv \langle n|U(g)|m\rangle, \qquad g \in G\,. \tag{3.1.1}$$

Then, we have

$$U(g)|n\rangle = \sum_m |m\rangle\langle m|U(g)|n\rangle = \sum_m |m\rangle D(g)_{m,n}\,. \tag{3.1.2}$$

Since the correspondence of $U(g)$ and $D(g)$ is straightforward, from now on we use the word "representation" interchangeably either designating $U(g)$ or $D(g)$.

When we say that two representations $D_1$ and $D_2$ are equivalent, we mean that the two representations are related to each other by a similarity transformation,

$$\exists \text{ a fixed } S : D_2(g) = SD_1(g)S^{-1}, \qquad \forall g \in G\,. \tag{3.1.3}$$

When we say that the representation $D(g)$ is reducible, we mean that $D(g)$ can be expressed in block diagonal form by a similarity transformation:

$$\exists S, \exists D', \forall g \in G : D'(g) = SD(g)S^{-1} = \begin{pmatrix} D'_1(g) & 0 \\ 0 & D'_2(g) \end{pmatrix}. \tag{3.1.4}$$

In this instance, we say that $D'$ is the direct sum of $D'_1$ and $D'_2$, and write

$$D' = D'_1 \oplus D'_2\,, \tag{3.1.5}$$

where $D'_1$ and $D'_2$ provide the mappings in their respective subspaces under the action of $U(g)$. When we say that the representation $D(g)$ is irreducible, we mean that $D(g)$ is not reducible.

### 3.1.2 Lie Groups

With these preparations, we move on to the discussion of Lie groups. A compact Lie group is the group of unitary operators whose elements are specified by finite numbers of a smooth continuous parameter and whose multiplication law depends on these smooth continuous parameters. By "compact" we mean that the volume of the parameter space is finite. Any representation of a compact Lie group is equivalent to a representation by a unitary operator. We write any group element which is smoothly connected to the identity element in the parameter space as

$$U(\varepsilon) = \exp[\mathrm{i}\varepsilon^\alpha T_\alpha]. \tag{3.1.6}$$

Here, $\varepsilon^\alpha$ $(\alpha = 1, \ldots, N)$ are $N$ real parameters and $T_\alpha$ $(\alpha = 1, \ldots, N)$ are the $N$ group generators which are linearly independent Hermitian operators defined on the Hilbert space. Linear combinations of the group generators, $\{\varepsilon^\alpha T_\alpha : \alpha = 1, \ldots, N, \varepsilon^\alpha \in \mathbb{R}\}$, span the vector space and $\{T_\alpha : \alpha = 1, \ldots, N\}$ are its basis vectors. Hereafter, the group generators are either arbitrary elements of this vector space or its basis vector $\{T_\alpha : \alpha = 1, \ldots, N\}$. We make the trivial statement that the $N$-dimensional vector space spanned by the group generators, $\{T_\alpha : \alpha = 1, \ldots, N\}$, and the infinite dimensional Hilbert space on which the unitary operator $U(\varepsilon)$ acts are two distinct vector spaces. We use the notation $G$ also for the $N$-dimensional vector space spanned by the group generators $\{T_\alpha : \alpha = 1, \ldots, N\}$ of the Lie group $G$.

The group generators $\{T_\alpha : \alpha = 1, \ldots, N\}$ not only span the $N$-dimensional vector space but also provide the commutators and the structure constants $C^{\gamma}_{\alpha\beta}$ from the group property of $G$, which almost completely determine the group structure. In order to clarify the latter point, we expand $U(\varepsilon)$, (3.1.6), to second order in $\varepsilon$:

$$U(\varepsilon) = 1 + \mathrm{i}\varepsilon^{\alpha} T_{\alpha} + \frac{1}{2}\varepsilon^{\alpha}\varepsilon^{\beta} T_{\alpha\beta} + O(\varepsilon^3)\,. \tag{3.1.7}$$

Here, $T_{\alpha}$ and $T_{\alpha\beta}$ are operators acting on the Hilbert space. From the unitarity of $U(\varepsilon)$,

$$U(\varepsilon)U^{\dagger}(\varepsilon) = 1\,, \tag{3.1.8}$$

we have

$$T^{\dagger}_{\alpha} = T_{\alpha}, \qquad T_{\alpha\beta} = T_{\beta\alpha}, \qquad \alpha, \beta = 1, \ldots, N\,. \tag{3.1.9}$$

The group multiplication law is expressed by

$$\begin{aligned} U(\varepsilon_1)U(\varepsilon_2) &= U(\varepsilon(\varepsilon_1, \varepsilon_2)) \\ &= \begin{cases} U(\varepsilon_1) & \text{if } \ \varepsilon_2 = 0 \\ U(\varepsilon_2) & \text{if } \ \varepsilon_1 = 0\,. \end{cases} \end{aligned} \tag{3.1.10}$$

From this, we obtain the following composition law:

$$\varepsilon^{\alpha}(\varepsilon_1, \varepsilon_2) = \varepsilon^{\alpha}_1 + \varepsilon^{\alpha}_2 + D^{\alpha}_{\beta\gamma}\varepsilon^{\beta}_1\varepsilon^{\gamma}_2 + O(\varepsilon^3)\,. \tag{3.1.11}$$

From the second order term in $\varepsilon$ in (3.10), we obtain

$$-T_{\alpha}T_{\beta} + T_{\alpha\beta} = \mathrm{i}D^{\gamma}_{\alpha\beta}T_{\gamma},$$

or,

$$T_{\alpha}T_{\beta} = T_{\alpha\beta} - \mathrm{i}D^{\gamma}_{\alpha\beta}T_{\gamma}\,. \tag{3.1.12}$$

From (3.1.9) and (3.1.12), we obtain the commutator of the generators as

$$[T_{\alpha}, T_{\beta}] = \mathrm{i}C^{\gamma}_{\alpha\beta}T_{\gamma}, \qquad \alpha, \beta, \gamma = 1, \ldots, N\,. \tag{3.1.13}$$

Here, we call $C^{\gamma}_{\alpha\beta}$ the structure constant which is given by

$$C^{\gamma}_{\alpha\beta} = -D^{\gamma}_{\alpha\beta} + D^{\gamma}_{\beta\alpha}\,, \tag{3.1.14}$$

with the following properties

(1) $C^{\gamma}_{\alpha\beta}$ is real and antisymmetric with respect to $\alpha$ and $\beta$,

$$C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha}\,, \tag{3.1.15}$$

(2) $C_{\alpha\beta}^{\gamma}$ satisfies the Jacobi identity,

$$C_{\alpha\beta}^{\gamma}C_{\gamma\delta}^{\varepsilon}+C_{\delta\alpha}^{\gamma}C_{\gamma\beta}^{\varepsilon}+C_{\beta\delta}^{\gamma}C_{\gamma\alpha}^{\varepsilon}=0\,. \tag{3.1.16}$$

When the Lie group $G$ originates from the internal symmetry of the internal degrees of freedom of quantum field theory, we have

$$U(g)\hat{\psi}_n(x)U^{-1}(g)=D_{n,m}(g)\hat{\psi}_m(x)\,. \tag{3.1.17}$$

The representation matrix $D(g)$ of the Lie group $G$ with respect to $\hat{\psi}_n(x)$ has the properties

$$D(g)D^{\dagger}(g)=1\,, \tag{3.1.18}$$

$$D(g_1)D(g_2)=D(g_1\circledcirc g_2)\,. \tag{3.1.19}$$

We parametrize $D(\varepsilon)$ similar to the linear unitary operator $U(\varepsilon)$ as

$$D(\varepsilon)=\exp[\mathrm{i}\varepsilon^{\alpha}t_{\alpha}]\,, \tag{3.1.20}$$

$$t_{\alpha}^{\dagger}=t_{\alpha}, \qquad \alpha=1,\ldots,N\,. \tag{3.1.21}$$

and express the group multiplication law (3.1.19) as

$$D(\varepsilon_1)D(\varepsilon_2)=D(\varepsilon(\varepsilon_1,\varepsilon_2))\,. \tag{3.1.22}$$

Based on the equivalence of the representations of $U(\varepsilon)$ and $D(\varepsilon)$, with the use of the composition law (3.1.11), we obtain the commutator of $t_{\alpha}$ by a similar argument leading to the commutator of the generator of the group (3.1.13) as

$$[t_{\alpha},t_{\beta}]=\mathrm{i}C_{\alpha\beta}^{\gamma}t_{\gamma}, \qquad \alpha,\beta,\gamma=1,\ldots,N\,. \tag{3.1.23}$$

We say that $t_{\alpha}$ is the realization of the generator $T_{\alpha}$ upon the field operator $\hat{\psi}_n(x)$. $\{t_{\alpha}:\alpha=1,\ldots,N\}$ span the $N$-dimensional vector space like the generator $T_{\alpha}$. This vector space is also designated by $G$.

We list the definitions of the invariant subalgebra, the invariant Abelian subalgebra, the simple algebra and the semisimple algebra:

(1) $\{t_X\}$, a subset of $\{t_{\alpha}\}_{\alpha=1}^{N}$, is called an invariant subalgebra if we have

$$\exists\{t_X\}\subset\{t_{\alpha}\}_{\alpha=1}^{N}:[t_{\alpha},t_X]=\text{linear combination of }t_X'\,. \tag{3.1.24}$$

(2) $\{t_X\}$, a subset of $\{t_{\alpha}\}_{\alpha=1}^{N}$, is called an invariant Abelian subalgebra if we have, on top of (3.1.24),

$$\forall t_X,t_Y\in\{t_X\}\subset\{t_{\alpha}\}_{\alpha=1}^{N}:[t_X,t_Y]=0\,. \tag{3.1.25}$$

(3) If the algebra does not have an invariant subalgebra, it is called a simple algebra.
(4) If the algebra does not have an invariant Abelian subalgebra, it is called a semisimple algebra. Note that the said algebra is allowed to have an invariant subalgebra as long as it is not Abelian.

Here, we state some theorems without the proof.

**Theorem**

(1) A Lie algebra can be decomposed into a semisimple Lie algebra $\{t_a\}$ and an invariant Abelian subalgebra $\{t_i\}$.
(2) A necessary and sufficient condition for the Lie algebra $\{t_\alpha\}_{\alpha=1}^N$ to be semisimple is that the matrix $\{g_{ab}\}_{a,b=1}^N$ defined below is positive definite:

$$g_{ab} \equiv -C^{\mathrm{d}}_{ac} C^{c}_{bd} \,. \tag{3.1.26}$$

(3) The semisimple Lie algebra $\{t_a\}$ can be decomposed into the direct sum of a simple Lie algebra $\{t^{(n)}_\xi\}_\xi$, $n = 1, \ldots,$

$$\{t_a\} = \sqcup_n \left\{t^{(n)}_\xi\right\}_\xi \,, \tag{3.1.27}$$

$$\left[t^{(n)}_\xi, t^{(n')}_{\xi'}\right] = \delta_{n,n'} \mathrm{i} C^{(n),\xi''}_{\xi\xi'} t^{(n)}_{\xi''} \,. \tag{3.1.28}$$

(Here, the simple Lie algebra $\{t^{(n)}_\xi\}_\xi$ does not posses an invariant subalgebra for each $n$.)

Since, in the discussion from the next section onward, semisimple Lie algebras play a major role, we present some characteristics of semisimple Lie algebras. If the Lie group $G$ is semisimple, we note that $\{g_{ab}\}_{a,b=1}^N$, defined by (3.1.26) in Theorem (2), is a real, symmetric and positive definite matrix. Hence, we can define $\{g^{1/2}\}$ and $\{g^{-1/2}\}$. We define $\tilde{C}_{abc}$ and $\tilde{t}_a$ by

$$\tilde{C}_{abc} = \left(g^{-1/2}\right)_{aa'} \left(g^{-\frac{1}{2}}\right)_{bb'} \left(g^{+\frac{1}{2}}\right)_{cc'} C^{c'}_{a'b'} \,, \tag{3.1.29}$$

$$\tilde{t}_a = \left(g^{-1/2}\right)_{aa'} t_{a'} \,. \tag{3.1.30}$$

From (3.1.23), (3.1.29) and (3.1.30), we obtain the commutator

$$[\tilde{t}_a, \tilde{t}_b] = \mathrm{i}\tilde{C}_{abc}\tilde{t}_c, \qquad a, b, c = 1, \ldots, N \,. \tag{3.1.31}$$

We observe that

(a) From (3.1.15) and (3.1.16), $\tilde{C}_{abc}$ is real and completely antisymmetric with respect to $a$, $b$ and $c$.

$$\tilde{C}_{abc} = -\tilde{C}_{bac}, \qquad (\text{from } (3.1.15))\,, \tag{3.1.32a}$$

$$= -\tilde{C}_{acb}, \qquad (\text{from } (3.1.16)). \tag{3.1.32b}$$

(b) $\tilde{C}_{abc}$ satisfy the Jacobi identity,

$$\tilde{C}_{abc}\tilde{C}_{cde} + \tilde{C}_{dac}\tilde{C}_{cbe} + \tilde{C}_{bdc}\tilde{C}_{cae} = 0\,. \tag{3.1.33}$$

If the Lie group is simple, by an appropriate normalization of the generator $\{T_a\}_{a=1}^{N}$, we have

$$\tilde{g}_{ab} \equiv -\tilde{C}_{acd}\tilde{C}_{bdc} \tag{3.1.34}$$

$$= \delta_{ab}\,. \tag{3.1.35}$$

If the Lie group is semisimple, by an appropriate normalization of the generator $\{T_a\}_{a=1}^{N}$, we have the block diagonal identity,

$$\tilde{g}_{ab} = \begin{pmatrix} \sim 1 & & & \\ & \sim 1 & & \\ & & \sim 1 & \\ & & & \sim 1 \end{pmatrix}. \tag{3.1.36}$$

By the introduction of the "metric tensor" $\{g_{ab}\}$ of the internal degrees of freedom, we could change the upper index to the lower index as in (3.1.29), and could show the complete antisymmetry of $\tilde{C}_{abc}$. From this point onward, we drop the tilde, "$\sim$".

Next, we define the adjoint representation of $G$, which plays an important role in the non-Abelian gauge field theory,

$$(t_{\alpha}^{\mathrm{adj}})_{\beta\gamma} = -\mathrm{i}C_{\alpha\beta\gamma}, \qquad \alpha, \beta, \gamma = 1, \ldots, N\,. \tag{3.1.37}$$

The adjoint representation is an imaginary and antisymmetric (and hence Hermitian) representation. The indices $(\beta, \gamma)$ of the internal degrees of freedom run from 1 to $N$, just like the index $\alpha$ of the group generator, $\{T_\alpha\}_{\alpha=1}^{N}$.

For future reference, we list the following Lie groups.

(1) $SU(N)$: all unitary $N \times N$ matrices with determinant $= 1$.
(2) $SO(N)$: all orthogonal $N \times N$ matrices with determinant $= 1$.
(3) $Sp(N)$
(4) Five series of exceptional groups.

## 3.2 Non-Abelian Gauge Field Theory

In this section, based on Weyl's gauge principle, we discuss the classical theory of the Abelian ($U(1)$) gauge field and the non-Abelian ($SU(2)$ or $SU(3)$) gauge field.

In Sect. 3.2.1, from Weyl's gauge principle, we discuss the relationship between the global $U(1)$ invariance of the matter system (charge conservation law) and the local $U(1)$ invariance of the matter-gauge system. The local extension of the global $U(1)$ invariance of the matter system is accomplished by

(1) the introduction of a $U(1)$ gauge field $A_\mu(x)$ and the replacement of the derivative $\partial_\mu\psi(x)$ in the matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))$$

with covariant derivative $D_\mu\psi(x)$,

$$\partial_\mu\psi(x) \to D_\mu\psi(x) \equiv (\partial_\mu + \mathrm{i}qA_\mu(x))\psi(x),$$

and

(2) the requirement that the covariant derivative $D_\mu\psi(x)$ transforms exactly like the matter field $\psi(x)$ under the local $U(1)$ phase transformation of $\psi(x)$.

Next, we motivate the local extension of global $SU(2)$ invariance (the total isospin conservation law) of nuclear physics. In Sect. 3.2.2, by taking the semisimple Lie group as the gauge group $G$, we construct a non-Abelian gauge field with the use of Weyl's gauge principle. In Sect. 3.2.3, we compare the Abelian gauge field and the non-Abelian gauge field from the standpoints of linearity vs. nonlinearity of the equation of motion and neutrality vs. non-neutrality of the gauge field. In Sect. 3.2.4, we present examples of the non-Abelian gauge field by taking an $SU(2)$-isospin gauge group and an $SU(3)$-color gauge group as the gauge group $G$.

### 3.2.1 Gauge Principle: "$U(1) \Rightarrow SU(2)$ Isospin"

The electrodynamics of James Clark Maxwell is described with the use of the 4-vector potential $A_\mu(x)$ by the total Lagrangian density

$$\begin{aligned}\mathcal{L}_{\text{tot}} = {} & \mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x)) + \mathcal{L}_{\text{int}}(\psi(x), A_\mu(x)) \\ & + \mathcal{L}_{\text{gauge}}(A_\mu(x), \partial_\nu A_\mu(x))\,. \end{aligned} \tag{3.2.1}$$

This system is invariant under the following local $U(1)$ transformation,

$$A_\mu(x) \to A'_\mu(x) \equiv A_\mu(x) - \partial_\mu\varepsilon(x)\,, \tag{3.2.2}$$

$$\psi(x) \to \psi'(x) \equiv \exp[\mathrm{i}q\varepsilon(x)]\psi(x)\,. \tag{3.2.3}$$

The interaction Lagrangian density $\mathcal{L}_{\text{int}}(\psi(x), A_\mu(x))$ of this system is the $j\cdot A$ coupling produced by the minimal substitution rule in $\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))$,

$$\partial_\mu\psi(x) \to D_\mu\psi(x) \equiv (\partial_\mu + \mathrm{i}qA_\mu(x))\psi(x)\,. \tag{3.2.4}$$

The transformation property of the covariant derivative $D_\mu\psi(x)$ under the local $U(1)$ transformation, (3.2.2) and (3.2.3), is given by

$$\begin{aligned} D_\mu\psi(x) \to (D_\mu\psi(x))' &= (\partial_\mu + \mathrm{i}qA'_\mu(x))\psi'(x) \\ &= (\partial_\mu + \mathrm{i}qA_\mu(x) - \mathrm{i}q\partial_\mu\varepsilon(x))\{\exp[\mathrm{i}q\varepsilon(x)]\psi(x)\} \\ &= \exp[\mathrm{i}q\varepsilon(x)]D_\mu\psi(x)\,, \end{aligned} \tag{3.2.5}$$

namely, $D_\mu\psi(x)$ transforms exactly like $\psi(x)$, (3.2.3).

The physical meaning of this local $U(1)$ invariance lies in its weaker version, global $U(1)$ invariance,

$$\varepsilon(x) = \varepsilon, \qquad \text{space-time independent constant.}$$

Global $U(1)$ invariance of the matter system under the global $U(1)$ transformation of $\psi(x)$,

$$\psi(x) \to \psi''(x) = \exp[\mathrm{i}q\varepsilon]\psi(x), \qquad \varepsilon = \text{constant}, \tag{3.2.6}$$

in its infinitesimal version results in

$$\frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial\psi(x)}\delta\psi(x) + \frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial(\partial_\mu\psi(x))}\delta(\partial_\mu\psi(x)) = 0\,, \tag{3.2.7}$$

with

$$\delta\psi(x) = \mathrm{i}q\varepsilon\cdot\psi(x)\,. \tag{3.2.8}$$

With the use of the Euler–Lagrange equation of motion, we obtain the current conservation law,

$$\partial_\mu j^\mu_{\text{matter}}(x) = 0\,, \tag{3.2.9}$$

$$\varepsilon\cdot j^\mu_{\text{matter}}(x) = \frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial(\partial_\mu\psi(x))}\delta\psi(x)\,, \tag{3.2.10}$$

which in its integrated form becomes charge conservation law,

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_{\text{matter}}(t) = 0\,, \tag{3.2.11}$$

$$Q_{\text{matter}}(t) = \int \mathrm{d}^3\boldsymbol{x} \cdot j^0_{\text{matter}}(t, \boldsymbol{x})\,. \tag{3.2.12}$$

Weyl's gauge principle considers the analysis backwards. The extension of the "charge conserving" global $U(1)$ invariance, (3.2.7) and (3.2.8), of the matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu \psi(x))$$

to the local $U(1)$ invariance necessitates

(1) the introduction of the $U(1)$ gauge field $A_\mu(x)$, and the replacement of the derivative $\partial_\mu \psi(x)$ in the matter field Lagrangian density with the covariant derivative $D_\mu \psi(x)$ defined by (3.2.4);
(2) the requirement that the covariant derivative $D_\mu \psi(x)$ transforms exactly like the matter field $\psi(x)$ under the local $U(1)$ phase transformation of $\psi(x)$, (3.2.3).

From the requirement (2), the transformation property of the $U(1)$ gauge field $A_\mu(x)$, (3.2.2), follows immediately. Also, from the requirement (2), the local $U(1)$ invariance of $\mathcal{L}_{\text{matter}}(\psi(x), D_\mu \psi(x))$ is self-evident. In order to give the dynamical content to the gauge field, we introduce the field strength tensor $F_{\mu\nu}(x)$ to the gauge field Lagrangian density $\mathcal{L}_{\text{gauge}}(A_\mu(x), \partial_\nu A_\mu(x))$ by the following trick,

$$[D_\mu, D_\nu]\psi(x) = \mathrm{i}q(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))\psi(x) \equiv \mathrm{i}qF_{\mu\nu}(x)\psi(x)\,. \tag{3.2.13}$$

The fact that the field strength tensor $F_{\mu\nu}(x)$, defined by (3.2.13), is a locally $U(1)$ invariant quantity follows from the transformation law of $A_\mu(x)$, (3.2.2),

$$\begin{aligned} F_{\mu\nu}(x) \to F'_{\mu\nu}(x) &= \partial_\mu A'_\nu(x) - \partial_\nu A'_\mu(x) \\ &= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = F_{\mu\nu}(x)\,. \end{aligned} \tag{3.2.14}$$

We choose as the gauge field Lagrangian density $\mathcal{L}_{\text{gauge}}(A_\mu(x), \partial_\nu A_\mu(x))$,

$$\mathcal{L}_{\text{gauge}}(A_\mu(x), \partial_\nu A_\mu(x)) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)\,. \tag{3.2.15}$$

In this manner, we obtain the total Lagrangian density of the matter-gauge system which is locally $U(1)$ invariant as,

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{matter}}(\psi(x), D_\mu \psi(x)) + \mathcal{L}_{\text{gauge}}(F_{\mu\nu}(x))\,. \tag{3.2.16}$$

The interaction Lagrangian density $\mathcal{L}_{\text{int}}(\psi(x), A_\mu(x))$ of (3.2.1) is given by

$$\begin{aligned} \mathcal{L}_{\text{int}}(\psi(x), A_\mu(x)) = \mathcal{L}_{\text{matter}}&(\psi(x), D_\mu \psi(x)) \\ &-\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu \psi(x))\,, \end{aligned} \tag{3.2.17}$$

and is the universal coupling generated by Weyl's gauge principle. In this way, as a result of the local extension of global $U(1)$ invariance, we derive the electrodynamics of J.C. Maxwell from the charge conservation law, (3.2.11) and (3.2.12), or from the current conservation law, (3.2.9) and (3.2.10).

Next, we note "total isospin conservation" in nuclear physics. This is the natural consequence of the global $SU(2)$ isospin invariance of the nuclear force. From the standpoint of Weyl's gauge principle, we consider this global $SU(2)$ isospin invariance of the nuclear force. We raise the following two questions.

(1) What is the local $SU(2)$ isospin invariance?
(2) What are the corresponding gauge field, its transformation law and the form of the universal interaction?

We make a table:

| | | |
|---|---|---|
| Gauge group | $U(1)$ | $SU(2)$ |
| Global invariance | Charge conservation | Total isospin conservation |
| Local invariance | Electrodynamics | ? |

C.N. Yang and R.L. Mills explored this "?" and the corresponding gauge field is called the Yang–Mills gauge field. R. Utiyama constructed a non-Abelian gauge field theory based on his version of the gauge principle with a (semisimple) Lie group as the gauge group. S.L. Glashow and M. Gell-Mann analyzed non-Abelian gauge field theory including the analysis of the Lie group. There is also a gauge principle due to J.J. Sakurai.

The gauge principle we have discussed so far is Weyl's gauge principle and its extension. There is another kind of gauge principle due to Klein, Kaluza and Fock from which charge conservation law and total isospin conservation law are also discussed. The modern interpretation of the latter gauge principle is given by R. Jackiw.

In the next section, based on Weyl's gauge principle, we discuss the construction of the non-Abelian gauge field in general.

### 3.2.2 Non-Abelian Gauge Field Theory

We let the semisimple Lie group $G$ be the gauge group. We let the representation of $G$ in the Hilbert space be $U(g)$, and its matrix representation on the field operator $\hat{\psi}_n(x)$ in the internal space be $D(g)$,

$$U(g)\hat{\psi}_n(x)U^{-1}(g) = D_{n,m}(g)\hat{\psi}_m(x), \qquad g \in G\,. \tag{3.2.18}$$

For the element $g_\varepsilon \in G$ continuously connected to the identity of $G$ by the parameter $\{\varepsilon_\alpha\}_{\alpha=1}^{N}$, we have

$$U(g_\varepsilon) = \exp[\mathrm{i}\varepsilon_\alpha T_\alpha] = 1 + \mathrm{i}\varepsilon_\alpha T_\alpha + \cdots\,, \tag{3.2.19}$$

$T_\alpha$ : generator of Lie group $G$,

$$D(g_\varepsilon) = \exp[\mathrm{i}\varepsilon_\alpha t_\alpha] = 1 + \mathrm{i}\varepsilon_\alpha t_\alpha + \cdots\,, \tag{3.2.20}$$

$t_\alpha$ : realization of $T_\alpha$ on the multiplet $\hat{\psi}_n(x)$,

$$[T_\alpha, T_\beta] = \mathrm{i}C_{\alpha\beta\gamma}T_\gamma\,, \tag{3.2.21}$$

$$[t_\alpha, t_\beta] = \mathrm{i}C_{\alpha\beta\gamma}t_\gamma\,. \tag{3.2.22}$$

We assume that the action functional $I_{\mathrm{matter}}[\psi_n]$ of the matter field Lagrangian density $\mathcal{L}_{\mathrm{matter}}(\psi_n(x), \partial_\mu\psi_n(x))$ given by

$$I_{\mathrm{matter}}[\psi_n] \equiv \int \mathrm{d}^4x \mathcal{L}_{\mathrm{matter}}(\psi_n(x), \partial_\mu\psi_n(x)) \tag{3.2.23}$$

is invariant under the global $G$ transformation,

$$\delta\psi_n(x) = \mathrm{i}\varepsilon_\alpha(t_\alpha)_{n,m}\psi_m(x), \qquad \varepsilon_\alpha = \text{constant}. \tag{3.2.24}$$

Namely, we have

$$\begin{aligned}&\frac{\partial\mathcal{L}_{\mathrm{matter}}(\psi_n(x), \partial_\mu\psi_n(x))}{\partial\psi_n(x)}\delta\psi_n(x)\\ &\qquad + \frac{\partial\mathcal{L}_{\mathrm{matter}}(\psi_n(x), \partial_\mu\psi_n(x))}{\partial(\partial_\mu\psi_n(x))}\delta(\partial_\mu\psi_n(x)) = 0\,.\end{aligned} \tag{3.2.25}$$

With the use of the Euler–Lagrange equation of motion,

$$\frac{\partial\mathcal{L}_{\mathrm{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial\psi_n(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\mathrm{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial(\partial_\mu\psi_n(x))}\right) = 0\,, \tag{3.2.26}$$

we have current conservation law and charge conservation law.

$$\partial_\mu j^\mu_{\alpha,\mathrm{matter}}(x) = 0, \qquad \alpha = 1, \ldots, N\,, \tag{3.2.27a}$$

$$\varepsilon_\alpha j^\mu_{\alpha,\mathrm{matter}}(x) = \frac{\partial\mathcal{L}_{\mathrm{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial(\partial_\mu\psi_n(x))}\delta\psi_n(x)\,, \tag{3.2.27b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}Q^{\mathrm{matter}}_\alpha(t) = 0, \qquad \alpha = 1, \ldots, N\,, \tag{3.2.28a}$$

$$Q_\alpha^{\text{matter}}(t) = \int \mathrm{d}^3\boldsymbol{x} j^0_{\alpha,\text{matter}}(t,\boldsymbol{x}), \qquad \alpha = 1, \ldots, N\,. \tag{3.2.28b}$$

Invoking Weyl's gauge principle, we extend the global $G$ invariance of the matter system to the local $G$ invariance of the matter system under the local $G$ phase transformation,

$$\delta\psi_n(x) = \mathrm{i}\varepsilon_\alpha(x)(t_\alpha)_{n,m}\psi_m(x)\,. \tag{3.2.29a}$$

We demand the followings:

(1) the introduction of the non-Abelian gauge field $A_{\alpha\mu}(x)$ and the replacement of the derivative $\partial_\mu\psi_n(x)$ in the matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))$$

with the covariant derivative $(D_\mu\psi(x))_n$,

$$\partial_\mu\psi_n(x) \to (D_\mu\psi(x))_n \equiv (\partial_\mu\delta_{n,m} + \mathrm{i}(t_\gamma)_{n,m}A_{\gamma\mu}(x))\psi_m(x)\,, \tag{3.2.30}$$

(2) the requirement that the covariant derivative $(D_\mu\psi(x))_n$ transforms exactly like the matter field $\psi_n(x)$ under the local $G$ phase transformation of $\psi_n(x)$, (3.2.29a),

$$\delta(D_\mu\psi(x))_n = \mathrm{i}\varepsilon_\alpha(x)(t_\alpha)_{n,m}(D_\mu\psi(x))_m\,, \tag{3.2.31a}$$

where $t_\gamma$ is the realization of the generator $T_\gamma$ upon the multiplet $\psi_n(x)$.

From (3.2.29a) and (3.2.31a), the infinitesimal transformation law of the non-Abelian gauge field $A_{\alpha\mu}(x)$ follows:

$$\delta A_{\alpha\mu}(x) = -\partial_\mu\varepsilon_\alpha(x) + \mathrm{i}\varepsilon_\beta(x)(t_\beta^{\text{adj}})_{\alpha\gamma}A_{\gamma\mu}(x) \tag{3.2.32a}$$

$$= -\partial_\mu\varepsilon_\alpha(x) + \varepsilon_\beta(x)C_{\beta\alpha\gamma}A_{\gamma\mu}(x)\,. \tag{3.2.32b}$$

Then, the local $G$ invariance of the gauged matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))$$

becomes self-evident as long as the ungauged matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))$$

is globally $G$ invariant. In order to provide the dynamical content to the non-Abelian gauge field $A_{\alpha\mu}(x)$, we introduce the field strength tensor $F_{\gamma\mu\nu}(x)$ by the following trick,

$$[D_\mu, D_\nu]\psi(x) \equiv \mathrm{i}(t_\gamma)F_{\gamma\mu\nu}(x)\psi(x)\,, \tag{3.2.33}$$

$$F_{\gamma\mu\nu}(x) = \partial_\mu A_{\gamma\nu}(x) - \partial_\nu A_{\gamma\mu}(x) - C_{\alpha\beta\gamma} A_{\alpha\mu}(x) A_{\beta\nu}(x) \,. \tag{3.2.34}$$

We can easily show that the field strength tensor $F_{\gamma\mu\nu}(x)$ undergoes a local $G$ rotation under the local $G$ transformations, (3.2.29a) and (3.2.32a), with the adjoint representation, (3.1.37).

$$\delta F_{\gamma\mu\nu}(x) = \mathrm{i}\varepsilon_\alpha(x)(t_\alpha^{\mathrm{adj}})_{\gamma\beta} F_{\beta\mu\nu}(x) \tag{3.2.35a}$$

$$= \varepsilon_\alpha(x) C_{\alpha\gamma\beta} F_{\beta\mu\nu}(x) \,. \tag{3.2.35b}$$

As the Lagrangian density of the non-Abelian gauge field $A_{\alpha\mu}(x)$, we choose

$$\mathcal{L}_{\mathrm{gauge}}(A_{\gamma\mu}(x), \partial_\nu A_{\gamma\mu}(x)) \equiv -\frac{1}{4} F_{\gamma\mu\nu}(x) F_\gamma^{\mu\nu}(x) \,. \tag{3.2.36}$$

The total Lagrangian density $\mathcal{L}_{\mathrm{total}}$ of the matter-gauge system is given by

$$\mathcal{L}_{\mathrm{total}} = \mathcal{L}_{\mathrm{matter}}(\psi(x), D_\mu \psi(x)) + \mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(x)) \,. \tag{3.2.37}$$

The interaction Lagrangian density $\mathcal{L}_{\mathrm{int}}$ consists of two parts due to the nonlinearity of the field strength tensor $F_{\gamma\mu\nu}(x)$ with respect to $A_{\gamma\mu}(x)$,

$$\begin{aligned} \mathcal{L}_{\mathrm{int}} = {} & \mathcal{L}_{\mathrm{matter}}(\psi(x), D_\mu \psi(x)) - \mathcal{L}_{\mathrm{matter}}(\psi(x), \partial_\mu \psi(x)) \\ & + \mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(x)) - \mathcal{L}_{\mathrm{gauge}}^{\mathrm{quad}}(F_{\gamma\mu\nu}(x)) \,, \end{aligned} \tag{3.2.38}$$

which provides a universal coupling just like the $U(1)$ gauge field theory. The conserved current $j^\mu_{\alpha,\mathrm{total}}(x)$ also consists of two parts,

$$j^\mu_{\alpha,\mathrm{total}}(x) \equiv j^\mu_{\alpha,\mathrm{matter}}(x) + j^\mu_{\alpha,\mathrm{gauge}}(x) \equiv \frac{\delta I_{\mathrm{total}}[\psi, A_{\alpha\mu}]}{\delta A_{\alpha\mu}(x)} \,, \tag{3.2.39a}$$

$$I_{\mathrm{total}}[\psi, A_{\alpha\mu}] = \int \mathrm{d}^4 x \{\mathcal{L}_{\mathrm{matter}}(\psi(x), D_\mu \psi(x)) + \mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(x))\}. \tag{3.2.39b}$$

The conserved charge $\{Q_\alpha^{\mathrm{total}}(t)\}_{\alpha=1}^N$ after extension to local $G$ invariance also consists of two parts,

$$\begin{aligned} Q_\alpha^{\mathrm{total}}(t) &= Q_\alpha^{\mathrm{matter}}(t) + Q_\alpha^{\mathrm{gauge}}(t) \\ &= \int \mathrm{d}^3 \boldsymbol{x} \{j^0_{\alpha,\mathrm{matter}}(t, \boldsymbol{x}) + j^0_{\alpha,\mathrm{gauge}}(t, \boldsymbol{x})\} \,. \end{aligned} \tag{3.2.40}$$

The gauged matter current $j^\mu_{\alpha,\mathrm{matter}}(x)$ of (3.2.39a) is not identical to the ungauged matter current $j^\mu_{\alpha,\mathrm{matter}}(x)$ of (3.2.27b):

$$\varepsilon_\alpha j^\mu_{\alpha,\mathrm{matter}}(x) \text{ of (3.2.27b)} = \frac{\partial \mathcal{L}_{\mathrm{matter}}(\psi(x), \partial_\mu \psi(x))}{\partial(\partial_\mu \psi_n(x))} \delta\psi_n(x)$$

whereas after local $G$ extension,

$$\begin{aligned}\varepsilon_\alpha j^\mu_{\alpha,\text{matter}}(x) \text{ of (3.2.39a)} &= \frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))}{\partial(D_\mu\psi_n(x))}\delta\psi_n(x) \\ &= \frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))}{\partial(D_\mu\psi(x))_n}\mathrm{i}\varepsilon_\alpha(t_\alpha)_{n,m}\psi_m(x) \\ &= \varepsilon_\alpha\frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))}{\partial(D_\nu\psi(x))_n}\frac{\partial(D_\nu\psi(x))_n}{\partial A_{\alpha\mu}(x)} \\ &= \varepsilon_\alpha\frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))}{\partial A_{\alpha\mu}(x)} \\ &= \varepsilon_\alpha\frac{\delta}{\delta A_{\alpha\mu}(x)}I_{\text{matter}}[\psi, D_\mu\psi]\,. \end{aligned} \tag{3.2.41}$$

Here, we note that $I_{\text{matter}}[\psi, D_\mu\psi]$ is not identical to the ungauged matter action functional $I_{\text{matter}}[\psi_n]$ given by (3.2.23), but is the gauged matter action functional defined by

$$I_{\text{matter}}[\psi, D_\mu\psi] \equiv \int \mathrm{d}^4x\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))\,. \tag{3.2.42}$$

We emphasize here that the conserved Noether current after extension of the global $G$ invariance to local $G$ invariance is not the gauged matter current $j^\mu_{\alpha,\text{matter}}(x)$ but the total current $j^\mu_{\alpha,\text{total}}(x)$, (3.2.39a). This has to do with the fact that the Lie group $G$ under consideration is non-Abelian and hence that the non-Abelian gauge field $A_{\alpha\mu}(x)$ is self-charged as opposed to the neutral $U(1)$ gauge field $A_\mu(x)$.

Before closing this section, we discuss the finite gauge transformation property of the non-Abelian gauge field $A_{\alpha\mu}(x)$. Under the finite local $G$ transformation of $\psi_n(x)$,

$$\psi_n(x) \to \psi'_n(x) = (\exp[\mathrm{i}\varepsilon_\alpha(x)t_\alpha])_{n,m}\psi_m(x)\,, \tag{3.2.29b}$$

we demand that the covariant derivative $D_\mu\psi(x)$ defined by (3.2.30) transforms exactly like $\psi(x)$,

$$\begin{aligned}D_\mu\psi(x) \to (D_\mu\psi(x))' &= (\partial_\mu + \mathrm{i}t_\gamma A'_{\gamma\mu}(x))\psi'(x) \\ &= \exp[\mathrm{i}\varepsilon_\alpha(x)t_\alpha]D_\mu\psi(x)\,. \end{aligned} \tag{3.2.31b}$$

From (3.2.31b), we obtain the following equation,

$$\begin{aligned}&\exp[-\mathrm{i}\varepsilon_\alpha(x)t_\alpha](\partial_\mu + \mathrm{i}t_\gamma A'_{\gamma\mu}(x))\exp[\mathrm{i}\varepsilon_\alpha(x)t_\alpha]\psi(x) \\ &\quad = (\partial_\mu + \mathrm{i}t_\gamma A_{\gamma\mu}(x))\psi(x).\end{aligned}$$

Cancelling the $\partial_\mu\psi(x)$ term on both sides of the above equation, and dividing by $\psi(x)$, we obtain

$$\exp[-\mathrm{i}\varepsilon_\alpha(x)t_\alpha](\partial_\mu \exp[\mathrm{i}\varepsilon_\alpha(x)t_\alpha]) + \exp[-\mathrm{i}\varepsilon_\alpha(x)t_\alpha](\mathrm{i}t_\gamma A'_{\gamma\mu}(x)) \times \exp[\mathrm{i}\varepsilon_\alpha(x)t_\alpha] = \mathrm{i}t_\gamma A_{\gamma\mu}(x).$$

Solving this equation for $t_\gamma A'_{\gamma\mu}(x)$, we finally obtain the finite gauge transformation law of $A'_{\gamma\mu}(x)$,

$$t_\gamma A'_{\gamma\mu}(x) = \exp[\mathrm{i}\varepsilon_\alpha(x)t_\alpha]\{t_\gamma A_{\gamma\mu}(x) + \exp[-\mathrm{i}\varepsilon_\beta(x)t_\beta](\mathrm{i}\partial_\mu \exp[\mathrm{i}\varepsilon_\beta(x)t_\beta])\} \times \exp[-\mathrm{i}\varepsilon_\alpha(x)t_\alpha]\,. \tag{3.2.32c}$$

At first sight, the finite gauge transformation law of $A'_{\gamma\mu}(x)$, (3.2.32c), may depend on the specific realization $\{t_\gamma\}_{\gamma=1}^N$ of the generator $\{T_\gamma\}_{\gamma=1}^N$ upon the multiplet $\psi(x)$. But, actually, $A'_{\gamma\mu}(x)$ transforms under the adjoint representation $\{t_\gamma^{\mathrm{adj}}\}_{\gamma=1}^N$. This point will become clear when we consider the infinitesimal version of (3.2.32c).

$$\begin{aligned} t_\gamma A'_{\gamma\mu}(x) &= t_\gamma A_{\gamma\mu}(x) + \mathrm{i}\varepsilon_\alpha(x)[t_\alpha, t_\beta]A_{\beta\mu}(x) - t_\gamma \partial_\mu \varepsilon_\gamma(x) \\ &= t_\gamma\{A_{\gamma\mu}(x) + \mathrm{i}\varepsilon_\alpha(x)(t_\alpha^{\mathrm{adj}})_{\gamma\beta}A_{\beta\mu}(x) - \partial_\mu\varepsilon_\gamma(x)\}. \end{aligned}$$

Multiplying $t_\gamma$ on both sides and taking the trace, we obtain,

$$\delta A_{\gamma\mu}(x) = -\partial_\mu\varepsilon_\gamma(x) + \mathrm{i}\varepsilon_\alpha(x)(t_\alpha^{\mathrm{adj}})_{\gamma\beta}A_{\beta\mu}(x),$$

which is (3.2.32a). Clearly, $A_{\gamma\mu}(x)$ transforms under the adjoint representation $\{t_\alpha^{\mathrm{adj}}\}_{\alpha=1}^N$ of $G$. For later convenience, we write (3.2.32a) as follows,

$$\begin{aligned} \delta A_{\gamma\mu}(x) &= -\partial_\mu\varepsilon_\gamma(x) + \varepsilon_\alpha(x)C_{\alpha\gamma\beta}A_{\beta\mu}(x) \\ &= -(\partial_\mu\varepsilon_\gamma(x) + \mathrm{i}(t_\beta^{\mathrm{adj}})_{\gamma\alpha}A_{\beta\mu}(x)\varepsilon_\alpha(x)) \\ &= -(\partial_\mu + \mathrm{i}t_\beta^{\mathrm{adj}}A_{\beta\mu}(x))_{\gamma\alpha}\varepsilon_\alpha(x) \\ &= -(D_\mu^{\mathrm{adj}}\varepsilon(x))_\gamma\,, \end{aligned} \tag{3.2.32d}$$

where we define

$$D_\mu^{\mathrm{adj}} = \partial_\mu + \mathrm{i}t_\beta^{\mathrm{adj}}A_{\beta\mu}(x)\,. \tag{3.2.32e}$$

### 3.2.3 Abelian Gauge Fields vs. Non-Abelian Gauge Fields

In the matter-gauge system which is locally $G$ invariant, due to the fact that $G$ is Non-Abelian, we found in Sect. 3.2.2 that the field strength tensor $F_{\gamma\mu\nu}(x)$ is nonlinear with respect to the non-Abelian gauge field $A_{\alpha\mu}(x)$ and that the interaction Lagrangian density $\mathcal{L}_{\mathrm{int}}$ which provides the universal coupling, the total conserved $G$-current $\{j^\mu_{\alpha,\mathrm{total}}(x)\}_{\alpha=1}^N$ and the total conserved $G$-charge

$\{Q_\alpha^{\text{total}}(t)\}_{\alpha=1}^{N}$ all consist of a matter part and a gauge part. In Sect. 3.2.1, we found that the physical significance of the local $U(1)$ invariance lies in its weaker version, global $U(1)$ invariance, namely, the charge conservation law. In the present section, we explore the implication of global $G$ invariance of the matter-gauge system which is locally $G$-invariant.

There follows a table of the global $U(1)$ transformation law and the global $G$ transformation law.

| Global U(1) transformation law | Global G transformation law |
|---|---|
| $\delta\psi_n(x) = \mathrm{i}\varepsilon q_n\psi_n(x)$, charged | $\delta\psi_n(x) = \mathrm{i}\varepsilon_\alpha(t_\alpha)_{n,m}\psi_m(x)$, charged |
| $\delta A_\mu(x) = 0$, neutral | $\delta A_{\alpha\mu}(x) = \mathrm{i}\varepsilon_\beta(t_\beta^{\text{adj}})_{\alpha\gamma}A_{\gamma\mu}(x)$, charged |

(3.2.43)

In the global transformation of the internal symmetry, (3.2.43), the matter fields $\psi_n(x)$ which have the group charge undergo a global ($U(1)$ or $G$) rotation. As for the gauge fields, $A_\mu(x)$ and $A_{\alpha\mu}(x)$, the Abelian gauge field $A_\mu(x)$ remains unchanged under the global $U(1)$ transformation while the non-Abelian gauge field $A_{\alpha\mu}(x)$ undergoes a global $G$ rotation under the global $G$ transformation. Hence, the Abelian gauge field $A_\mu(x)$ is $U(1)$-neutral while the non-Abelian gauge field $A_{\alpha\mu}(x)$ is $G$-charged. The field strength tensors, $F_{\mu\nu}(x)$ and $F_{\alpha\mu\nu}(x)$, behave like $A_\mu(x)$ and $A_{\alpha\mu}(x)$, under the global $U(1)$ and $G$ transformations. The field strength tensor $F_{\mu\nu}(x)$ is $U(1)$-neutral, while the field strength tensor $F_{\alpha\mu\nu}(x)$ is $G$-charged, which originate from their linearity and nonlinearity in $A_\mu(x)$ and $A_{\alpha\mu}(x)$, respectively.

| Global $U(1)$ transformation law | Global $G$ transformation law |
|---|---|
| $\delta F_{\mu\nu}(x) = 0$, neutral | $\delta F_{\alpha\mu\nu}(x) = \mathrm{i}\varepsilon_\beta(t_\beta^{\text{adj}})_{\alpha\gamma}F_{\gamma\mu\nu}(x)$, charged |

(3.2.44)

When we write down the Euler–Lagrange equation of motion for each case, the linearity and the nonlinearity with respect to the gauge fields become clear.

| Abelian $U(1)$ gauge field | Non-Abelian $G$gauge field | |
|---|---|---|
| $\partial_\nu F^{\nu\mu}(x) = j^\mu_{\text{matter}}(x)$, linear | $D_\nu^{\text{adj}}F_\alpha^{\nu\mu}(x) = j^\mu_{\alpha,\text{matter}}(x)$, nonlinear | (3.2.45) |

From the antisymmetry of the field strength tensor with respect to the Lorentz indices, $\mu$ and $\nu$, we have the following current conservation as an identity.

| Abelian $U(1)$ gauge field | Non-Abelian $G$ gauge field | |
|---|---|---|
| $\partial_\mu j^\mu_{\text{matter}}(x) = 0$ | $D_\mu^{\text{adj}} j^\mu_{\alpha,\text{matter}}(x) = 0$ | (3.2.46) |

Here, we have

$$D_\mu^{\text{adj}} = \partial_\mu + \mathrm{i}t_\gamma^{\text{adj}}A_{\gamma\mu}(x)\,. \tag{3.2.47}$$

As a result of the extension to local ($U(1)$ or $G$) invariance, in the case of the Abelian $U(1)$ gauge field, due to the neutrality of $A_\mu(x)$, the matter current $j^\mu_{\text{matter}}(x)$ alone (which originates from the global $U(1)$ invariance) is conserved, while in the case of the non-Abelian $G$ gauge field, due to the $G$-charge of $A_{\alpha\mu}(x)$, the gauged matter current $j^\mu_{\alpha,\text{matter}}(x)$ alone (which originates from the local $G$ invariance) is not conserved, but the sum with the gauge current $j^\mu_{\alpha,\text{gauge}}(x)$ which originates from the self-interaction of the non-Abelian gauge field $A_{\alpha\mu}(x)$ is conserved at the expense of loss of covariance. A similar situation exists for the charge conservation law.

| Abelian $U(1)$ gauge field | Non-Abelian $G$ gauge field | |
|---|---|---|
| $\frac{\mathrm{d}}{\mathrm{d}t}Q^{\text{matter}}(t)=0$ | $\frac{\mathrm{d}}{\mathrm{d}t}Q^{\text{total}}_\alpha(t)=0$ | |
| $Q^{\text{matter}}(t)=\int \mathrm{d}^3\boldsymbol{x}\cdot j^0_{\text{matter}}(t,\boldsymbol{x})$ | $Q^{\text{total}}_\alpha(t)=\int \mathrm{d}^3\boldsymbol{x}\cdot j^0_{\alpha,\text{total}}(t,\boldsymbol{x})$ | (3.2.48) |

### 3.2.4 Examples

In this section, we discuss examples of the extension of global $G$ invariance of the matter system to local $G$ invariance of the matter-gauge system. For this purpose, we first establish several notations.

**Boson Field**: $\phi_i(x)$ realization $(\theta_\alpha)_{i,j}$ $[\theta_\alpha,\theta_\beta]=\mathrm{i}C_{\alpha\beta\gamma}\theta_\gamma$.
Infinitesimal local $G$ transformation.

$$\delta\phi_i(x)=\mathrm{i}\varepsilon_\alpha(x)(\theta_\alpha)_{i,j}\phi_j(x)\,. \tag{3.2.49a}$$

**Fermion Field**: $\psi_n(x)$ realization $(t_\alpha)_{n,m}$ $[t_\alpha,t_\beta]=\mathrm{i}C_{\alpha\beta\gamma}t_\gamma$.
Infinitesimal local $G$ transformation.

$$\delta\psi_n(x)=\mathrm{i}\varepsilon_\alpha(x)(t_\alpha)_{n,m}\psi_m(x)\,. \tag{3.2.49b}$$

**Gauge Field**: $A_{\alpha\mu}(x)$ realization $(t^{\text{adj}}_\alpha)_{\beta,\gamma}$ $[t^{\text{adj}}_\alpha,t^{\text{adj}}_\beta]=\mathrm{i}C_{\alpha\beta\gamma}t^{\text{adj}}_\gamma$.
Infinitesimal local $G$ transformation.

$$\delta A_{\alpha\mu}(x)=-\partial_\mu\varepsilon_\alpha(x)+\mathrm{i}\varepsilon_\beta(x)(t^{\text{adj}}_\beta)_{\alpha,\gamma}A_{\gamma\mu}(x)\,. \tag{3.2.49c}$$

**Covariant Derivative**

**Boson Field**:

$$(D_\mu\phi(x))_i=(\partial_\mu+\mathrm{i}\theta_\alpha A_{\alpha\mu}(x))_{i,j}\phi_j(x)\,. \tag{3.2.50a}$$

**Fermion Field**:

$$(D_\mu\psi(x))_n=(\partial_\mu+\mathrm{i}t_\alpha A_{\alpha\mu}(x))_{n,m}\psi_m(x)\,. \tag{3.2.50b}$$

**Gauge Field**:

$$(D_\mu^{\mathrm{adj}})_{\alpha,\beta} = (\partial_\mu + \mathrm{i}t_\gamma^{\mathrm{adj}} A_{\gamma\mu}(x))_{\alpha,\beta} \,. \tag{3.2.50c}$$

**Example 3.1.** Dirac particle (spin $\frac{1}{2}$) in interaction with a non-Abelian gauge field. We consider the Dirac Lagrangian density $\mathcal{L}_{\mathrm{Dirac}}(\psi(x), \partial_\mu\psi(x))$ which is globally $G$ invariant,

$$\mathcal{L}_{\mathrm{Dirac}}(\psi(x), \partial_\mu\psi(x)) = \bar{\psi}_n(x)\{(\mathrm{i}\gamma^\mu\partial_\mu - m)\delta_{n,,m}\}\psi_m(x) \,. \tag{3.2.51}$$

$G = SU(2)$: We assume that the Dirac particle $\psi_n(x)$ transforms under the fundamental representation of $SU(2)$:

$$t_\alpha = \frac{1}{2}\tau_\alpha, \qquad (t_\alpha^{\mathrm{adj}})_{\beta\gamma} = -\mathrm{i}\varepsilon_{\alpha\beta\gamma}, \qquad \alpha, \beta, \gamma = 1, 2, 3 \,; \tag{3.2.52}$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{3.2.53}$$

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As a result of the extension to local $SU(2)$ invariance, we obtain $\mathcal{L}_{\mathrm{tot}}$ as

$$\begin{aligned} \mathcal{L}_{\mathrm{tot}} &= -\frac{1}{4}(\partial_\mu A_{\alpha\nu}(x) - \partial_\nu A_{\alpha\mu} - \varepsilon_{\alpha\beta\gamma}A_{\beta\mu}(x)A_{\gamma\nu}(x))^2 \\ &\quad + \bar{\psi}_n(x)\Big\{\mathrm{i}\gamma^\mu\Big(\partial_\mu\delta_{n,m} + \mathrm{i}\big(\tfrac{1}{2}\tau_\alpha\big)_{n,m} A_{\alpha\mu}(x)\Big) - m\delta_{n,m}\Big\}\psi_m(x). \end{aligned} \tag{3.2.54}$$

Here, $SU(2)$ is called the isospin gauge group, and we have the following identification,

$$\begin{aligned} \psi_n(x) &: \quad \text{Nucleon field, (iso-doublet), } n = 1, 2 \,. \\ A_{\alpha\mu}(x) &: \quad \text{Yang–Mills gauge field, (iso-triplet), } \alpha = 1, 2, 3 \,. \end{aligned} \tag{3.2.55}$$

This non-Abelian gauge field was derived by C.N. Yang and R.L. Mills in 1954 as a result of the extension of total isospin conservation to local $SU(2)$ invariance. This is a local extension of the custom that we call a nucleon with isospin up a proton and a nucleon with isospin down a neutron.

$G = SU(3)$: We assume that the Dirac particle $\psi_n(x)$ transforms under the fundamental representation of $SU(3)$.

$$t_\alpha = \frac{1}{2}\lambda_\alpha, \qquad (t_\alpha^{\mathrm{adj}})_{\beta,\gamma} = -\mathrm{i}f_{\alpha\beta\gamma}, \qquad \alpha, \beta, \gamma = 1, \ldots, 8 \,; \tag{3.2.56}$$

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \lambda_2=\begin{pmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&0\\0&0&0\end{pmatrix},\quad \lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$

$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \lambda_5=\begin{pmatrix}0&0&-\mathrm{i}\\0&0&0\\\mathrm{i}&0&0\end{pmatrix},\quad \lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\lambda_7=\begin{pmatrix}0&0&0\\0&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix},\quad \lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}; \tag{3.2.57}$$

$$\lambda_0=\sqrt{\frac{2}{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

As a result of the extension to local $SU(3)$ invariance, we obtain $\mathcal{L}_{\mathrm{tot}}$ as

$$\begin{aligned}\mathcal{L}_{\mathrm{tot}}&=-\frac{1}{4}(\partial_\mu A_{\alpha\nu}(x)-\partial_\nu A_{\alpha\mu}(x)-f_{\alpha\beta\gamma}A_{\beta\mu}(x)A_{\gamma\nu}(x))^2\\&\quad+\bar{\psi}_n(x)\left\{\mathrm{i}\gamma^\mu\left(\partial_\mu\delta_{n,m}+\mathrm{i}\left(\tfrac{1}{2}\lambda_\alpha\right)_{n,m}A_{\alpha\mu}(x)\right)-m\delta_{n,m}\right\}\psi_m(x)\,.\end{aligned} \tag{3.2.58}$$

Here, $SU(3)$ is called the color gauge group, and we have the following identification,

$$\begin{array}{rll}\psi_n(x): & \text{quark (color triplet)}, & n=1,2,3.\\ A_{\alpha\mu}(x): & \text{gluon (color octet)}, & \alpha=1,\dots,8.\end{array} \tag{3.2.59}$$

This is the "*color-triplet quark color-octet gluon*" *model* proposed by H. Fritzsch, M. Gell-Mann and H. Leutwyler in 1972. As a predecessor to this model, we have the three-triplet model proposed by M.Y. Han and Y. Nambu. Both of these models are $SU(3)$ Yang–Mills gauge models which resolve the spin-statistics problem of the quark-ace model of Gell-Mann and Zweig proposed in 1964 by the *color-singlet hypothesis.*

**Example 3.2.** Klein–Gordon particle (spin 0) in interaction with a non-Abelian gauge field. We consider the real scalar Lagrangian density $\mathcal{L}^{\mathrm{real}}_{\mathrm{scalar}}$ $(\phi_\alpha(x),\partial_\mu\phi_\alpha(x))$ and the complex scalar Lagrangian density $\mathcal{L}^{\mathrm{complex}}_{\mathrm{scalar}}(\phi_i(x),$ $\partial_\mu\phi_i(x);\phi_i^\dagger(x),\partial_\mu\phi_i^\dagger(x))$ which are globally $G$ invariant,

$$\begin{aligned}\mathcal{L}^{\mathrm{real}}_{\mathrm{scalar}}(\phi(x),\partial_\mu\phi(x))&=\frac{1}{2}(\partial_\mu\phi_\alpha(x))^2-\frac{1}{2}m^2(\phi_\alpha(x)\phi_\alpha(x))\\&\quad-\lambda^2(\phi_\alpha(x)\phi_\alpha(x))^2\end{aligned} \tag{3.2.60}$$

$$\begin{aligned}\mathcal{L}^{\text{complex}}_{\text{scalar}}&(\phi(x),\partial_\mu\phi(x);\phi^\dagger(x),\partial_\mu\phi^\dagger(x))\\ &=\partial_\mu\phi_i^\dagger(x)\partial^\mu\phi_i(x)-m^2(\phi_i^\dagger(x)\phi_i(x))-\lambda^2(\phi_i^\dagger(x)\phi_i(x))^2\,. \end{aligned}\tag{3.2.61}$$

$G=SU(2)$: We assume that the real scalar particle $\phi_\alpha(x)$ (the complex scalar particle $\phi_i(x)$) transforms under the adjoint representation (the fundamental representation) of $SU(2)$:

$$(\theta_\alpha^{\text{adj}})_{\beta\gamma}=-\mathrm{i}\varepsilon_{\alpha\beta\gamma},\quad \alpha,\beta,\gamma=1,2,3,\quad \text{real scalar iso-triplet}\,,\tag{3.2.62}$$

$$\theta_\alpha=\frac{1}{2}\tau_\alpha,\quad \alpha=1,2,3,\quad \text{complex scalar iso-doublet.}\tag{3.2.63}$$

As a result of the extension to local $SU(2)$ invariance, we obtain $\mathcal{L}_{\text{tot}}$ as

$$\begin{aligned}\mathcal{L}_{\text{tot}}=&-\tfrac{1}{4}(\partial_\mu A_{\alpha\nu}(x)-\partial_\nu A_{\alpha\mu}(x)-\varepsilon_{\alpha\beta\gamma}A_{\beta\mu}(x)A_{\gamma\nu}(x))^2\\ &+\tfrac{1}{2}\left(\partial_\mu\phi_\alpha(x)-\varepsilon_{\alpha\beta\gamma}A_{\beta\mu}(x)\phi_\gamma(x)\right)^2\\ &-\tfrac{1}{2}m^2\left(\phi_\alpha(x)\phi_\alpha(x)\right)-\lambda^2(\phi_\alpha(x)\phi_\alpha(x))^2\,,\end{aligned}\tag{3.2.64}$$

$$\begin{aligned}\mathcal{L}_{\text{tot}}=&-\tfrac{1}{4}(\partial_\mu A_{\alpha\nu}(x)-\partial_\nu A_{\alpha\mu}(x)-\varepsilon_{\alpha\beta\gamma}A_{\beta\mu}(x)A_{\gamma\nu}(x))^2\\ &+\Big(\partial_\mu\phi_i(x)+\mathrm{i}\left(\tfrac{1}{2}\tau_\alpha\right)_{i,j}A_{\alpha\mu}(x)\phi_j(x)\Big)^\dagger\\ &\times\Big(\partial_\mu\phi_i(x)+\mathrm{i}\left(\tfrac{1}{2}\tau_\beta\right)_{i,k}A_{\beta\mu}(x)\phi_k(x)\Big)\\ &-m^2(\phi_i^\dagger(x)\phi_i(x))-\lambda^2(\phi_i^\dagger(x)\phi_i(x))^2\,.\end{aligned}\tag{3.2.65}$$

As a final example, we discuss the most general renormalizable Lagrangian density.

**Example 3.3.** Canonical locally $G$ invariant Lagrangian density. The most general Lagrangian density which is locally $G$ invariant and renormalizable by power counting, involving $\psi_n(x)$, $\phi_i(x)$ and $A_{\alpha\mu}(x)$, is given by

$$\begin{aligned}\mathcal{L}_{\text{tot}}=&\bar{\psi}_n(x)(\mathrm{i}\gamma^\mu D_\mu-m)_{n,m}\psi_m(x)+\tfrac{1}{2}(D_\mu\phi(x))_i(D^\mu\phi(x))_i\\ &-\tfrac{1}{4}F_{\alpha\mu\nu}(x)F_\alpha^{\mu\nu}(x)+\bar{\psi}_n(x)(\Gamma_i)_{n,m}\psi_m(x)\phi_i(x)\\ &+V(\phi_i(x))\,.\end{aligned}\tag{3.2.66}$$

(1) The $\phi_i(x)$ are real scalar fields. From the reality of the local $G$ transformation,

$$\delta\phi_i(x)=\mathrm{i}\varepsilon_\alpha(x)(\theta_\alpha)_{i,j}\phi_j(x)\,,\tag{3.2.49a}$$

we find that $\{\theta_\alpha\}_{\alpha=1}^N$ are imaginary and antisymmetric (and hence Hermitian) matrices,

$$\theta_\alpha^* = \theta_\alpha^{\mathrm{T}} = -\theta_\alpha, \qquad \alpha = 1, \ldots, N. \tag{3.2.67}$$

(2) $V(\phi(x))$ is a locally $G$ invariant quartic polynomial,

$$\frac{\partial V(\phi(x))}{\partial \phi_i(x)}(\theta_\alpha)_{i,j}\phi_j(x) = 0, \qquad \alpha = 1, \ldots, N. \tag{3.2.68}$$

(3) The mass term of $\psi_n(x)$ is invariant under the local $G$ transformation,

$$\delta\psi_n(x) = \mathrm{i}\varepsilon_\alpha(x)(t_\alpha)_{n,,m}\psi_m(x), \tag{3.2.49b}$$

i.e., we have

$$[t_\alpha, \gamma^0 m] = 0. \tag{3.2.69}$$

(4) The coupling of $\phi_i(x)$ and $\psi_n(x)$ is a gauge covariant Yukawa coupling under the local $G$ transformations, (3.2.49a) and (3.2.49b),

$$[t_\alpha, \gamma^0\Gamma_i] = \gamma^0\Gamma_j(\theta_\alpha)_{j,i} = -(\theta_\alpha)_{i,j}\gamma^0\Gamma_j. \tag{3.2.70}$$

(5) The total Lagrangian density $\mathcal{L}_{\mathrm{tot}}$, (3.2.66), is Hermitian, i.e., we have

$$V^* = V, \tag{3.2.71a}$$

$$m^\dagger = \gamma^0 m \gamma^0, \tag{3.2.71b}$$

$$\Gamma_i^\dagger = \gamma^0\Gamma_i\gamma^0. \tag{3.2.71c}$$

## 3.3 Path Integral Quantization of Gauge Fields

In this section, we discuss path integral quantization of the gauge field. When we apply the path integral representation of $\langle 0,\mathrm{out}|0,\mathrm{in}\rangle$ obtained in Chap. 2,

$$\langle 0,\mathrm{out}|0,\mathrm{in}\rangle = \int \mathcal{D}[\phi_i]\exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x))\right],$$

naively to path integral quantization of the gauge field, we obtain results which violate unitarity. The origin of this difficulty lies in the gauge invariance of the gauge field action functional $I_{\mathrm{gauge}}[A_{\alpha\mu}]$, i.e., the four-dimensional transversality of the kernel of the quadratic part of the gauge field action functional $I_{\mathrm{gauge}}[A_{\alpha\mu}]$. In other words, when we path-integrate along the gauge equivalent class, the functional integrand remains constant and we merely calculate the orbit volume $V_G$. We introduce the hypersurface (the gauge fixing condition)

$$F_\alpha(A_{\beta\mu}(x)) = 0, \qquad \alpha = 1, \ldots, N,$$

in the manifold of the gauge field which intersects with each gauge equivalent class only once and perform path integration and group integration on this hypersurface. We shall complete the path integral quantization of the gauge field and the separation of the orbit volume $V_G$ simultaneously (Sect. 3.3.1). This method is called the Faddeev–Popov method. Next, we generalize the gauge-fixing condition to the form

$$F_\alpha(A_{\beta\mu}(x)) = a_\alpha(x), \qquad \alpha = 1, \ldots, N,$$

with $a_\alpha(x)$ an arbitrary function independent of $A_{\alpha\mu}(x)$. From this consideration, we obtain the second formula of Faddeev–Popov and we arrive at the standard covariant gauge (Sect. 3.3.2). Roughly speaking, we introduce the Faddeev–Popov ghost (fictitious scalar fermion) only in the internal loop, which restores unitarity. Lastly, we discuss the various gauge-fixing conditions (Sect. 3.3.3), and the advantage of the axial gauge.

### 3.3.1 The First Faddeev–Popov Formula

In Chap. 2, we derived the path integral formula for the vacuum-to-vacuum transition amplitude $\langle 0,\text{out}|0,\text{in}\rangle$ for the nonsingular Lagrangian density $\mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x))$ as

$$\langle 0,\text{out}|0,\text{in}\rangle = \int \mathcal{D}[\phi] \exp\left[\mathrm{i}\int \mathrm{d}^4x \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x))\right]. \tag{3.3.1}$$

We cannot apply this path integral formula naively to non-Abelian gauge field theory. The kernel $K_{\alpha\mu,\beta\nu}(x-y)$ of the quadratic part of the action functional of the gauge field $A_{\gamma\mu}(x)$,

$$\begin{aligned} I_{\text{gauge}}[A_{\gamma\mu}] &= \int \mathrm{d}^4x \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) \\ &= -\frac{1}{4}\int \mathrm{d}^4x F_{\gamma\mu\nu}(x) F_\gamma^{\mu\nu}(x)\,, \end{aligned} \tag{3.3.2a}$$

$$F_{\gamma\mu\nu}(x) = \partial_\mu A_{\gamma\nu}(x) - \partial_\nu A_{\gamma\mu}(x) - C_{\alpha\beta\gamma} A_{\alpha\mu}(x) A_{\beta\nu}(x)\,, \tag{3.3.2b}$$

is given by

$$I_{\text{gauge}}^{\text{quad}}[A_{\gamma\mu}] = -\frac{1}{2}\iint \mathrm{d}^4x \mathrm{d}^4y A_\alpha^\mu(x) K_{\alpha\mu,\beta\nu}(x-y) A_\beta^\nu(y)\,, \tag{3.3.3a}$$

$$K_{\alpha\mu,\beta\nu}(x-y)=\delta_{\alpha\beta}(-\eta_{\mu\nu}\partial^2+\partial_\mu\partial_\nu)\delta^4(x-y), \quad \mu,\nu=0,1,2,3\,. \tag{3.3.3b}$$

$K_{\alpha\mu,\beta\nu}(x-y)$ is singular and noninvertible. We cannot define the "free" Green's function of $A_{\alpha\mu}(x)$ as it stands now, and we cannot apply the Feynman path integral formula, (3.3.1), for path integral quantization of the gauge field. As a matter of fact, this $K_{\alpha\mu,\beta\nu}(x-y)$ is the four-dimensional transverse projection operator for arbitrary $A_{\alpha\mu}(x)$. The origin of this calamity lies in the gauge invariance of the gauge field action functional $I_{\text{gauge}}[A_{\gamma\mu}]$ under the gauge transformation,

$$A_{\gamma\mu}(x)\to A^g_{\gamma\mu}(x)\,, \tag{3.3.4a}$$

with

$$t_\gamma A^g_{\gamma\mu}(x)=U(g(x))\{t_\gamma A_{\gamma\mu}(x) \\ +U^{-1}(g(x))[\mathrm{i}\partial_\mu U(g(x))]\}U^{-1}(g(x))\,, \tag{3.3.4b}$$

and

$$U(g(x))=\exp[\mathrm{i}g_\gamma(x)t_\gamma]\,. \tag{3.3.4c}$$

In other words, it originates from the fact that the gauge field action functional $I_{\text{gauge}}[A_{\gamma\mu}]$ is constant on the orbit of the gauge group $G$. Here, by the orbit we mean the gauge equivalent class,

$$\{A^g_{\gamma\mu}(x): A_{\gamma\mu}(x) \text{ fixed}, \forall g(x)\in G\}\,, \tag{3.3.5}$$

where $A^g_{\gamma\mu}(x)$ is defined by (3.3.4b). Hence, the naive expression for the vacuum-to-vacuum transition amplitude $\langle 0,\text{out}|0,\text{in}\rangle$ of the gauge field $A_{\gamma\mu}(x)$,

$$\langle 0,\text{out}|0,\text{in}\rangle \text{ “=” } \int \mathcal{D}[A_{\gamma\mu}]\exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\right]\,, \tag{3.3.6}$$

is proportional to the orbit volume $V_G$,

$$V_G=\int \mathrm{d}g(x)\,, \tag{3.3.7}$$

which is infinite and independent of $A_{\gamma\mu}(x)$. In order to accomplish path integral quantization of the gauge field $A_{\gamma\mu}(x)$ correctly, we must extract the orbit volume $V_G$ from the vacuum-to-vacuum transition amplitude,

$$\langle 0,\text{out}|0,\text{in}\rangle.$$

In the path integral

$$\int \mathcal{D}[A_{\gamma\mu}]$$

in (3.3.6), we should not path-integrate over all possible configurations of the gauge field, but we should path-integrate only over distinct orbits of the gauge field $A_{\gamma\mu}(x)$. For this purpose, we shall introduce the hypersurface (the gauge-fixing condition),

$$F_\alpha(A_{\gamma\mu}(x)) = 0, \qquad \alpha = 1, \ldots, N, \tag{3.3.8}$$

in the manifold of the gauge field $A_{\gamma\mu}(x)$, which intersects with each orbit only once. The statement that the hypersurface, (3.3.8), intersects with each orbit only once implies that

$$F_\alpha(A^g_{\gamma\mu}(x)) = 0, \qquad \alpha = 1, \ldots, N, \tag{3.3.9}$$

has the unique solution $g(x) \in G$ for arbitrary $A_{\gamma\mu}(x)$. In this sense, the hypersurface, (3.3.8), is called the gauge-fixing condition. Here, we recall group theory:

$$\text{(1)}\ \ g(x), g'(x) \in G \Rightarrow (gg')(x) \in G. \tag{3.3.10a}$$

$$\text{(2)}\ \ U(g(x))U(g'(x)) = U((gg')(x)). \tag{3.3.10b}$$

$$\text{(3)}\ \ \mathrm{d}g'(x) = \mathrm{d}(gg')(x), \qquad \text{invariant Hurwitz measure.} \tag{3.3.10c}$$

We parametrize $U(g(x))$ in the neighborhood of the identity element of $G$ as

$$U(g(x)) = 1 + \mathrm{i}t_\gamma\varepsilon_\gamma(x) + O(\varepsilon^2), \tag{3.3.11a}$$

$$\prod_x \mathrm{d}g(x) = \prod_{\alpha,x} \mathrm{d}\varepsilon_\alpha(x) \quad \text{for } g(x) \approx \text{Identity}. \tag{3.3.11b}$$

With these preparations, we define the *Faddeev–Popov determinant* $\Delta_F[A_{\gamma\mu}]$ by

$$\Delta_F[A_{\gamma\mu}] \int \prod_x \mathrm{d}g(x) \prod_{\alpha,x} \delta(F_\alpha(A^g_{\gamma\mu}(x))) = 1. \tag{3.3.12}$$

We first show that $\Delta_F[A_{\gamma\mu}]$ is gauge invariant under the gauge transformation

$$A_{\gamma\mu}(x) \to A^g_{\gamma\mu}(x). \tag{3.3.4a}$$

$$\begin{aligned}
(\Delta_F[A^g_{\gamma\mu}])^{-1} &= \int \prod_x \mathrm{d}g'(x) \prod_{\alpha,x} \delta(F_\alpha(A^{gg'}_{\gamma\mu})) \\
&= \int \prod_x \mathrm{d}(gg')(x) \prod_{\alpha,x} \delta(F_\alpha(A^{gg'}_{\gamma\mu})) \\
&= \int \prod_x \mathrm{d}g''(x) \prod_{\alpha,x} \delta(F_\alpha(A^{g''}_{\gamma\mu})) \\
&= (\Delta_F[A_{\gamma\mu}])^{-1},
\end{aligned}$$

i.e., we have

$$\Delta_F[A^g_{\gamma\mu}] = \Delta_F[A_{\gamma\mu}]\,. \tag{3.3.13}$$

We substitute the defining equation of the Faddeev–Popov determinant into the functional integrand on the right-hand side of the naive expression, (3.3.6), for the vacuum-to-vacuum transition amplitude $\langle 0,\text{out}|0,\text{in}\rangle$.

$$\begin{aligned}\langle 0,\text{out}|0,\text{in}\rangle \text{ ``}=\text{''} &\int \prod_x \mathrm{d}g(x)\mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]\prod_{\alpha,x}\delta(F_\alpha(A^g_{\gamma\mu}(x)))\\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\right]\,. \end{aligned}\tag{3.3.14}$$

In the integrand of (3.3.14), we perform the following gauge transformation,

$$A_{\gamma\mu}(x) \to A^{g^{-1}}_{\gamma\mu}(x)\,. \tag{3.3.15}$$

Under this gauge transformation, we have

$$\begin{aligned}&\mathcal{D}[A_{\gamma\mu}] \to \mathcal{D}[A_{\gamma\mu}], \quad \Delta_F[A_{\gamma\mu}] \to \Delta_F[A_{\gamma\mu}],\\ &\delta(F_\alpha(A^g_{\gamma\mu}(x))) \to \delta(F_\alpha(A_{\gamma\mu}(x)))\,,\\ &\int \mathrm{d}^4x\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) \to \int \mathrm{d}^4x\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\,.\end{aligned}\tag{3.3.16}$$

Since the value of the functional integral remains unchanged under a change of function variable, from (3.3.14) and (3.3.16), we have

$$\begin{aligned}\langle 0,\text{out}|0,\text{in}\rangle \text{ ``}=\text{''} &\int \prod_x \mathrm{d}g(x)\int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]\prod_{\alpha,x}\delta(F_\alpha(A_{\gamma\mu}(x)))\\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\right]\\ = \; & V_G\int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]\prod_{\alpha,x}\delta(F_\alpha(A_{\gamma\mu}(x)))\\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\right]\,. \end{aligned}\tag{3.3.17}$$

In this manner, we extract the orbit volume $V_G$, which is infinite and is independent of $A_{\gamma\mu}(x)$, and at the expense of the introduction of $\Delta_F[A_{\gamma\mu}]$, we perform the path integral of the gauge field

$$\int \mathcal{D}[A_{\gamma\mu}]$$

on the hypersurface

$$F_\alpha(A_{\gamma\mu}(x)) = 0, \qquad \alpha = 1, \ldots, N\,, \tag{3.3.8}$$

which intersects with each orbit of the gauge field only once. From now on, we drop $V_G$ and write the vacuum-to-vacuum transition amplitude as

$$\begin{aligned}\langle 0, \mathrm{out}|0, \mathrm{in}\rangle_F = &\int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]\prod_{\alpha,x}\delta(F_\alpha(A_{\gamma\mu}(x)))\\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(x))\right]\,,\end{aligned} \tag{3.3.18}$$

under the gauge-fixing condition, (3.3.8).

In (3.3.18), noting that the Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$ gets multiplied by

$$\prod_{\alpha,x}\delta(F_\alpha(A_{\gamma\mu}(x))),$$

we calculate $\Delta_F[A_{\gamma\mu}]$ only for $A_{\gamma\mu}(x)$ which satisfies the gauge-fixing condition, (3.3.8). Mimicking the parametrization of $U(g(x))$ in the neighborhood of the identity element of $G$, (3.3.11a) and (3.3.11b), we parametrize $F_\alpha(A^g_{\gamma\mu}(x))$ as

$$\begin{aligned}F_\alpha(A^g_{\gamma\mu}(x)) = &F_\alpha(A_{\gamma\mu}(x))\\ &+\sum_{\beta=1}^{N}\int \mathrm{d}^4y M^F_{\alpha x,\beta y}(A_{\gamma\mu})\varepsilon_\beta(y) + O(\varepsilon^2)\,,\end{aligned} \tag{3.3.19}$$

$$\prod_x \mathrm{d}g(x) = \prod_{\alpha,x}\mathrm{d}\varepsilon_\alpha(x).$$

The matrix $M^F_{\alpha x,\beta y}(A_{\gamma\mu})$ is the kernel of the linear response with respect to $\varepsilon_\beta(y)$ of the gauge-fixing condition under the infinitesimal gauge transformation

$$\delta A^g_{\gamma\mu}(x) = -(D^{\mathrm{adj}}_\mu\varepsilon(x))_\gamma\,. \tag{3.3.20}$$

Thus, from (3.3.19) and (3.3.20), we have

$$\begin{aligned}M^F_{\alpha x,\beta y}(A_{\gamma\mu}) &= \frac{\delta F_\alpha(A^g_{\gamma\mu}(x))}{\delta\varepsilon_\beta(y)}|_{g=1}\\ &= \frac{\delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}(-D^{\mathrm{adj}}_\mu)_{\gamma\beta}\delta^4(x-y)\\ &\equiv M^F_{\alpha,\beta}(A_{\gamma\mu}(x))\delta^4(x-y)\,,\end{aligned} \tag{3.3.21}$$

with

$$(D_\mu^{\text{adj}})_{\alpha\beta} = \partial_\mu \delta_{\alpha\beta} + C_{\alpha\beta\gamma} A_{\gamma\mu}(x).$$

For those $A_{\gamma\mu}(x)$ which satisfy the gauge-fixing condition, (3.3.8), we have

$$\begin{aligned}(\Delta_F[A_{\gamma\mu}])^{-1} &= \int \prod_{\alpha,x} \mathrm{d}\varepsilon_\alpha(x) \prod_{\alpha,x} \delta(M^F_{\alpha,\beta}(A_{\gamma\mu}(x))\varepsilon_\beta(x)) \\ &= (\mathrm{Det} M^F(A_{\gamma\mu}))^{-1}\,,\end{aligned}$$

i.e., we have

$$\Delta_F[A_{\gamma\mu}] = \mathrm{Det} M^F(A_{\gamma\mu})\,. \tag{3.3.22}$$

We introduce the Faddeev–Popov ghost (fictitious scalar fermion), $\bar{c}_\alpha(x)$ and $c_\beta(x)$, in order to exponentiate the Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$. Then, the determinant $\Delta_F[A_{\gamma\mu}]$ can be written as

$$\begin{aligned}\Delta_F[A_{\gamma\mu}] &= \mathrm{Det} M^F(A_{\gamma\mu}) \\ &= \int \mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta] \exp\left[\mathrm{i}\int \mathrm{d}^4x \bar{c}_\alpha(x) M^F_{\alpha,\beta}(A_{\gamma\mu}(x)) c_\beta(x)\right].\end{aligned} \tag{3.3.23}$$

Thus, we obtain the first Faddeev–Popov formula:

$$\begin{aligned}\langle 0, \mathrm{out}|0, \mathrm{in}\rangle_F = &\int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \prod_{\alpha,x} \delta(F_\alpha(A_{\gamma\mu}(x))) \\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4x \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\right]\end{aligned} \tag{3.3.24a}$$

$$\begin{aligned}= &\int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta] \prod_{\alpha,x} \delta(F_\alpha(A_{\gamma\mu}(x))) \\ &\times \exp\Big[\mathrm{i}\int \mathrm{d}^4x \{\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) \\ &+ \bar{c}_\alpha(x) M^F_{\alpha,\beta}(A_{\gamma\mu}(x)) c_\beta(x)\}\Big]\,.\end{aligned} \tag{3.3.24b}$$

The Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$ plays the role of restoring the unitarity.

### 3.3.2 The Second Faddeev–Popov Formula

We now generalize the gauge-fixing condition, (3.3.8):

$$F_\alpha(A_{\gamma\mu}(x)) = a_\alpha(x), \quad \alpha = 1,\dots,N; \quad a_\alpha(x) \text{ independent of } A_{\alpha\mu}(x)\,. \tag{3.3.25}$$

According to the first Faddeev–Popov formula, (3.3.24a), we have

$$\begin{aligned}\mathcal{Z}_F(a(x)) &\equiv \langle 0, \text{out}|0, \text{in}\rangle_{F,a} \\ &= \int \mathcal{D}[A_{\gamma\mu}]\varDelta_F[A_{\gamma\mu}] \times \prod_{\alpha,x} \delta(F_\alpha(A_{\gamma\mu}(x)) - a_\alpha(x)) \\ &\qquad \times \exp\left[\mathrm{i}\int \mathrm{d}^4x \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\right]\,.\end{aligned} \tag{3.3.26}$$

We perform the nonlinear infinitesimal gauge transformation $g_0$, in the functional integrand of (3.3.26), parametrized by $\lambda_\beta(y)$,

$$\varepsilon_\alpha(x; M^F(A_{\gamma\mu})) = (M^F(A_{\gamma\mu}))^{-1}_{\alpha x,\beta y}\lambda_\beta(y)\,, \tag{3.3.27}$$

where $\lambda_\beta(y)$ is an arbitrary infinitesimal function, independent of $A_{\alpha\mu}(x)$. Under this $g_0$, we have the following three statements.

(1) The gauge invariance of the gauge field action functional,

$$\int \mathrm{d}^4x \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))$$

is invariant under $g_0$.

(2) The gauge invariance of the integration measure,

$$\mathcal{D}[A_{\gamma\mu}]\varDelta_F[A_{\gamma\mu}] \tag{3.3.28}$$

is invariant measure under $g_0$. (For the proof of this statement, see Appendix 4.)

(3) The gauge-fixing condition $F_\alpha(A_{\gamma\mu}(x))$ gets transformed into the following,

$$F_\alpha(A^{g_0}_{\gamma\mu}(x)) = F_\alpha(A_{\gamma\mu}(x)) + \lambda_\alpha(x) + O(\lambda^2_\alpha(x)). \tag{3.3.29}$$

Since the value of the functional integral remains unchanged under a change of function variable, the value of $\mathcal{Z}_F(a(x))$ remains unchanged under the nonlinear gauge transformation $g_0$. When we choose the infinitesimal parameter $\lambda_\alpha(x)$ as

$$\lambda_\alpha(x) = \delta a_\alpha(x), \qquad \alpha = 1,\dots,N\,, \tag{3.3.30}$$

we have from the above-stated (1), (2) and (3),

$$\mathcal{Z}_F(a(x)) = \mathcal{Z}_F(a(x) + \delta a(x)) \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}a(x)}\mathcal{Z}_F(a(x)) = 0\,. \tag{3.3.31}$$

Namely, $\mathcal{Z}_F(a(x))$ is independent of $a(x)$. Introducing an arbitrary weighting functional $H[a_\alpha(x)]$ for $\mathcal{Z}_F(a(x))$, and path-integrating with respect to $a_\alpha(x)$, we obtain the weighted $\mathcal{Z}_F(a(x))$:

$$\begin{aligned}\mathcal{Z}_F &\equiv \int \prod_{\alpha,x} \mathrm{d}a_\alpha(x) H[a_\alpha(x)]\mathcal{Z}_F(a(x)) \\ &= \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]H[F_\alpha(A_{\gamma\mu}(x))] \\ &\quad \times \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\right].\end{aligned} \tag{3.3.32}$$

As the weighting functional, we choose the quasi-Gaussian functional

$$H[a_\alpha(x)] = \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x a_\alpha^2(x)\right]. \tag{3.3.33}$$

Then, we obtain as $\mathcal{Z}_F$,

$$\begin{aligned}\mathcal{Z}_F &= \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]\exp\left[\mathrm{i}\int \mathrm{d}^4x\{\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) - \tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}(x))\}\right] \\ &= \int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[\mathrm{i}\int \mathrm{d}^4x\{\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) \right. \\ &\quad \left. -\tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}(x)) + \bar{c}_\alpha(x)M^F_{\alpha,\beta}(A_{\gamma\mu}(x))c_\beta(x)\}\right].\end{aligned} \tag{3.3.34}$$

Thus, we obtain the second Faddeev–Popov formula:

$$\langle 0, \text{out}|0, \text{in}\rangle_F = \int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta]\exp[\mathrm{i}I_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta]]\,, \tag{3.3.35a}$$

with the effective action functional $I_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta]$,

$$I_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta] = \int \mathrm{d}^4x\mathcal{L}_{\text{eff}}\,, \tag{3.3.35b}$$

where the effective Lagrangian density $\mathcal{L}_{\text{eff}}$ is given by

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) - \frac{1}{2}F_\alpha^2(A_{\gamma\mu}(x)) + \bar{c}_\alpha(x)M^F_{\alpha,\beta}(A_{\gamma\mu}(x))c_\beta(x)\,. \tag{3.3.35c}$$

There are other methods to arrive at this second formula, due to Fradkin and Tyutin, and Vassiliev.

### 3.3.3 Choice of Gauge-Fixing Condition

We consider the following gauge-fixing conditions.

(1) Axial gauge

$$F_\alpha(A_{\gamma\mu}(x)) = n^\mu A_{\alpha\mu}(x) = 0, \qquad \alpha = 1, \ldots, N\,, \tag{3.3.36a}$$

$$n^\mu = (0;0,0,1), \qquad n^2 = n^\mu n_\mu = -1, \qquad nk = n^\mu k_\mu = -k_3\,, \tag{3.3.36b}$$

(2) Landau gauge

$$F_\alpha(A_{\gamma\mu}(x)) = \partial^\mu A_{\alpha\mu}(x) = 0, \qquad \alpha = 1, \ldots, N\,, \tag{3.3.37}$$

(3) Covariant gauge

$$F_\alpha(A_{\gamma\mu}(x)) = \sqrt{\xi}\partial^\mu A_{\alpha\mu}(x), \quad \alpha = 1, \ldots, N, \quad 0 < \xi < \infty\,, \tag{3.3.38}$$

and obtain the Green's functions $D^{(A)}_{\alpha\mu,\beta\nu}(x-y)$ of the gauge field $A_{\gamma\mu}(x)$ and $D^{(C)}_{\alpha,\beta}(x-y)$ of the Faddeev–Popov ghost fields, $\bar{c}_\alpha(x)$ and $c_\beta(x)$, and the effective interaction Lagrangian density $\mathcal{L}^{\text{int}}_{\text{eff}}$.

**(1) Axial Gauge**: We use the first Faddeev–Popov formula, (3.3.24a). The kernel $K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$ of the quadratic part of the gauge field action functional $I_{\text{gauge}}[A_{\gamma\mu}]$ in the axial gauge, (3.3.36a) and (3.3.36b), is given by

$$I^{\text{quad}}_{\text{gauge}}[A_{\gamma\mu}] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y A^\mu_\alpha(x) K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y) A^\nu_\beta(y)$$

with

$$K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y) = \delta_{\alpha\beta}(-\eta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu)\delta^4(x-y) \tag{3.3.39a}$$

where

$$\mu, \nu = 0, 1, 2; \qquad \alpha, \beta = 1, \ldots, N\,. \tag{3.3.39b}$$

At first sight, this axial gauge kernel $K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$ appears to be a non-invertible kernel, (3.3.3b). Since the Lorentz indices, $\mu$ and $\nu$, run through 0, 1, 2 only, this axial gauge kernel $K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$ is invertible, contrary to (3.3.3b). This has to do with the fact that the third spatial component of the gauge field $A_{\gamma\mu}(x)$ gets killed by the gauge-fixing condition, (3.3.36a) and (3.3.36b).

The "free" Green's function $D^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$ of the gauge field in the axial gauge satisfies the following equation:

$$\delta_{\alpha\beta}(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)D^{(\text{axial})}_{\beta\nu,\gamma\lambda}(x-y) = \delta_{\alpha\gamma}\eta^\mu_\lambda\delta^4(x-y)\,. \tag{3.3.40}$$

Upon Fourier transforming the "free" Green's function $D^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$,

$$\begin{aligned} D^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y) &= \delta_{\alpha\beta}D^{(\text{axial})}_{\mu,\nu}(x-y) \\ &= \delta_{\alpha\beta}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\exp[-\mathrm{i}k(x-y)]D^{(\text{axial})}_{\mu,\nu}(k)\,, \end{aligned} \tag{3.3.41}$$

we have the momentum space Green's function $D^{(\text{axial})}_{\mu,\nu}(k)$ satisfying

$$(-\eta^{\mu\nu}k^2 + k^\mu k^\nu)D^{(\text{axial})}_{\nu\lambda}(k) = \eta^\mu_\lambda\,. \tag{3.3.42a}$$

In order to satisfy the axial gauge condition (3.3.36a) automatically, we introduce the transverse projection operator $\Lambda_{\mu\nu}(n)$ with respect to $n^\mu$

$$\Lambda_{\mu\nu}(n) = \eta_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} = \eta_{\mu\nu} + n_\mu n_\nu\,, \tag{3.3.43a}$$

with

$$n\Lambda(n) = \Lambda(n)n = 0, \qquad \Lambda^2(n) = \Lambda(n)\,, \tag{3.3.43b}$$

and write the momentum space Green's function $D^{(\text{axial})}_{\nu\lambda}(k)$ as,

$$D^{(\text{axial})}_{\nu\lambda}(k) = \Lambda_{\nu\sigma}(n)(\eta^{\sigma\rho}A(k^2) + k^\sigma k^\rho B(k^2))\Lambda_{\rho\lambda}(n)\,. \tag{3.3.44}$$

Left-multiplying $\Lambda_{\tau\mu}(n)$ in (3.3.42a), we have

$$\Lambda_{\tau\mu}(n)(-\eta^{\mu\nu}k^2 + k^\mu k^\nu)D^{(\text{axial})}_{\nu\lambda}(k) = \Lambda_{\tau\lambda}(n)\,. \tag{3.3.42b}$$

Substituting (3.3.44) into (3.3.42b) and making use of the identities

$$(k\Lambda(n))_\mu = (\Lambda(n)k)_\mu = k_\mu - \frac{(nk)}{n^2}n_\mu\,, \tag{3.3.45a}$$

$$k\Lambda(n)k = k^2 - \frac{(nk)^2}{n^2}\,, \tag{3.3.45b}$$

we have

$$\begin{aligned} &(-k^2A(k^2))\Lambda_{\tau\lambda}(n) + \left(A(k^2) - \frac{(nk)^2}{n^2}B(k^2)\right) \\ &\qquad \times(\Lambda(n)k)_\tau(k\Lambda(n))_\lambda = \Lambda_{\tau\lambda}(n)\,. \end{aligned} \tag{3.3.46}$$

From this, we obtain $A(k^2)$, $B(k^2)$ and $D^{(\text{axial})}_{\nu\lambda}(k)$ as

$$A(k^2) = -\frac{1}{k^2}, \qquad B(k^2) = \frac{n^2}{(nk)^2}A(k^2) = -\frac{1}{k^2}\frac{n^2}{(nk)^2}, \tag{3.3.47}$$

$$\begin{aligned}
&D^{(\text{axial})}_{\nu\lambda}(k) \\
&\quad = -\frac{1}{k^2}\left\{\eta_{\nu\lambda} - \frac{n_\nu n_\lambda}{n^2} + \frac{n^2}{(nk)^2}\left(k_\nu - \frac{(nk)}{n^2}n_\nu\right)\right. \\
&\qquad \left. \times \left(k_\lambda - \frac{(nk)}{n^2}n_\lambda\right)\right\} &(3.3.48\text{a}) \\
&\quad = -\frac{1}{k^2}\left\{\eta_{\nu\lambda} + \frac{n^2}{(nk)^2}k_\nu k_\lambda - \frac{1}{(nk)}n_\nu k_\lambda - \frac{1}{(nk)}k_\nu n_\lambda\right\} &(3.3.48\text{b}) \\
&\quad = -\frac{1}{k^2}\left\{\eta_{\nu\lambda} - \frac{k_\nu k_\lambda}{k_3^2} + \frac{n_\nu k_\lambda}{k_3} + \frac{k_\nu n_\lambda}{k_3}\right\}, &(3.3.48\text{c})
\end{aligned}$$

where in (3.3.48c) for the first time in this derivation, the following explicit representations for the axial gauge are used,

$$n^2 = -1, \qquad nk = -k_3\,. \tag{3.3.36b}$$

Next, we demonstrate that the Faddeev–Popov ghost does not show up in the axial gauge, (3.3.36a) and (3.3.36b). The kernel of the Faddeev–Popov ghost Lagrangian density is given by

$$\begin{aligned}
M^{(\text{axial})}_{\alpha x,\beta y}(A_{\gamma\mu}) &= \frac{\delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}(-D^{\text{adj}}_\mu)_{\gamma\beta}\delta^4(x-y) \\
&= -n^\mu(\delta_{\alpha\beta}\partial_\mu + C_{\alpha\beta\gamma}A_{\gamma\mu}(x))\delta^4(x-y) \\
&= \delta_{\alpha\beta}\partial_3\delta^4(x-y)\,,
\end{aligned} \tag{3.3.49}$$

which is independent of $A_{\gamma\mu}(x)$. Hence, the Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$ is an infinite number, independent of the gauge field $A_{\gamma\mu}(x)$, and the Faddeev–Popov ghost does not show up.

In this way we see that we can carry out canonical quantization of the non-Abelian gauge field theory in the axial gauge, starting out with the separation of the dynamical variable and the constraint variable. Thus, we can apply the phase space path integral formula developed in Chap. 2 with the minor modification of the insertion of the gauge-fixing condition

$$\prod_{\alpha,x}\delta(n^\mu A_{\alpha\mu}(x))$$

in the functional integrand. In the phase space path integral formula, with the use of the constraint equation, we perform the momentum integration and obtain

$$\langle 0,\mathrm{out}|0,\mathrm{in}\rangle = \int \mathcal{D}[A_{\gamma\mu}]\prod_{\alpha,x}\delta(n^\mu A_{\alpha\mu}(x)) \times \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(x))\right]. \tag{3.3.50}$$

In (3.3.50), the manifest covariance is miserably destroyed. This point will be overcome by the gauge transformation from the axial gauge to some covariant gauge, say the Landau gauge, and the Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$ will show up again as the Jacobian of the change of the function variable (the gauge transformation) in the delta function,

$$F_\alpha^{(\mathrm{axial})}(A_{\gamma\mu}(x)) \to F_\alpha^{(\mathrm{covariant})}(A_{\gamma\mu}(x)). \tag{3.3.51}$$

Equation (3.3.50) can be rewritten as the first Faddeev–Popov formula, (3.3.24a) and (3.3.24b).

The effective interaction Lagrangian density $\mathcal{L}_{\mathrm{eff}}^{\mathrm{int}}$ is given by

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{int}} = \mathcal{L}_{\mathrm{eff}} - \mathcal{L}_{\mathrm{eff}}^{\mathrm{quad}}. \tag{3.3.52}$$

**(2) Landau Gauge**: We use the first Faddeev–Popov formula, (3.3.24a) and (3.3.24b). The kernel $K_{\alpha\mu,\beta\nu}^{(\mathrm{Landau})}(x-y)$ of the quadratic part of the gauge field action functional $I_{\mathrm{gauge}}[A_{\gamma\mu}]$ in the Landau gauge, (3.3.37),

$$I_{\mathrm{gauge}}^{\mathrm{quad}}[A_{\gamma\mu}] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4yA_\alpha^\mu(x)K_{\alpha\mu,\beta\nu}^{(\mathrm{Landau})}(x-y)A_\beta^\nu(y)$$

is given by

$$K_{\alpha\mu,\beta\nu}^{(\mathrm{Landau})}(x-y) = \delta_{\alpha\beta}(-\eta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu)\delta^4(x-y), \tag{3.3.53a}$$

$$\mu,\nu = 0,1,2,3; \qquad \alpha,\beta = 1,\ldots,N. \tag{3.3.53b}$$

$K_{\alpha\mu,\beta\nu}^{(\mathrm{Landau})}(x-y)$ is a four-dimensional transverse projection operator and hence is not invertible. This difficulty is overcome by the introduction of the four-dimensional transverse projection operator $\Lambda_{\mathrm{T}}(k)$.

The "free" Green's function $D_{\alpha\mu,\beta\nu}^{(\mathrm{Landau})}(x-y)$ of the gauge field in the Landau gauge satisfies the equation

$$\delta_{\alpha\beta}(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)D_{\beta\nu,\gamma\lambda}^{(\mathrm{Landau})}(x-y) = \delta_{\alpha\gamma}\eta_\lambda^\mu\delta^4(x-y). \tag{3.3.54}$$

Upon Fourier transforming the "free" Green's function $D^{(\text{Landau})}_{\alpha\mu,\beta\nu}(x-y)$,

$$\begin{aligned} D^{(\text{Landau})}_{\alpha\mu,\beta\nu}(x-y) &= \delta_{\alpha\beta} D^{(\text{Landau})}_{\mu,\nu}(x-y) \\ &= \delta_{\alpha\beta} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp[-\mathrm{i}k(x-y)] D^{(\text{Landau})}_{\mu\nu}(k) , \end{aligned} \tag{3.3.55}$$

we have the momentum space Green's function $D^{(\text{Landau})}_{\mu\nu}(k)$ satisfying

$$(-\eta^{\mu\nu} k^2 + k^\mu k^\nu) D^{(\text{Landau})}_{\nu\lambda}(k) = \eta^\mu_\lambda . \tag{3.3.56a}$$

In order to satisfy the four-dimensional transversality, (3.3.37), automatically, we introduce the four-dimensional transverse projection operator $\Lambda_{\mathrm{T}}(k)$ for an arbitrary 4-vector $k^\mu$ by

$$\Lambda_{\mathrm{T}\ \mu\nu}(k) = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} , \tag{3.3.57a}$$

$$k\Lambda_{\mathrm{T}}(k) = \Lambda_{\mathrm{T}}(k) k = 0, \qquad \Lambda^2_{\mathrm{T}}(k) = \Lambda_{\mathrm{T}}(k) , \tag{3.3.57b}$$

and write $D^{(\text{Landau})}_{\nu\lambda}(k)$ as

$$D^{(\text{Landau})}_{\nu\lambda}(k) = \Lambda_{\mathrm{T}\ \nu\lambda}(k) A(k^2) . \tag{3.3.58}$$

Left-multiplying $\Lambda_{\mathrm{T}\ \tau\mu}(k)$ in (3.3.56a), we have

$$\Lambda_{\mathrm{T}\ \tau\mu}(k)(-k^2)\Lambda^{\mu\nu}_{\mathrm{T}}(k) D^{(\text{Landau})}_{\nu\lambda}(k) = \Lambda_{\mathrm{T}\ \tau\lambda}(k) . \tag{3.3.56b}$$

Substituting (3.3.58) into the left-hand side of (3.3.56b), and noting the idempotence of $\Lambda_{\mathrm{T}}(k)$, we have

$$(-k^2 A(k^2))\Lambda_{\mathrm{T}\ \tau\lambda}(k) = \Lambda_{\mathrm{T}\ \tau\lambda}(k) . \tag{3.3.59}$$

From (3.3.59), we obtain

$$A(k^2) = -\frac{1}{k^2} , \tag{3.3.60}$$

$$D^{(\text{Landau})}_{\nu\lambda}(k) = -\frac{1}{k^2}\left(\eta_{\nu\lambda} - \frac{k_\nu k_\lambda}{k^2}\right) . \tag{3.3.61}$$

We now calculate the kernel $M^{(\text{Landau})}_{\alpha x,\beta y}(A_{\gamma\mu})$ of the Faddeev–Popov ghost in the Landau gauge.

$$\begin{aligned}
M^{(\text{Landau})}_{\alpha x,\beta y}(A_{\gamma\mu}) &= \frac{\delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}(-D^{\text{adj}}_\mu)_{\gamma\beta}\delta^4(x-y) \\
&= -\partial^\mu(\partial_\mu\delta_{\alpha\beta} + C_{\alpha\beta\gamma}A_{\gamma\mu}(x))\delta^4(x-y) \\
&\equiv M^{\text{L}}_{\alpha,\beta}(A_{\gamma\mu}(x))\delta^4(x-y)\,. \qquad (3.3.62)
\end{aligned}$$

The Faddeev–Popov ghost part of $I_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta]$ is given by

$$\begin{aligned}
I^{\text{ghost}}_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta] &= \int \mathrm{d}^4x \bar{c}_\alpha(x) M^{\text{L}}_{\alpha,\beta}(A_{\gamma\mu}(x))c_\beta(x) \qquad (3.3.63) \\
&= \int \mathrm{d}^4x \partial^\mu \bar{c}_\alpha(x)(\partial_\mu c_\alpha(x) + C_{\alpha\beta\gamma}A_{\gamma\mu}(x)c_\beta(x))\,.
\end{aligned}$$

From this, we find that the Faddeev–Popov ghost in the Landau gauge is a massless scalar fermion and that the Green's function $D^{(C)}_{\text{L}\ \alpha,\beta}(x-y)$ is given by

$$D^{(C)}_{\text{L}\ \alpha,\beta}(x-y) = \delta_{\alpha\beta}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\exp[-\mathrm{i}k(x-y)]\frac{1}{k^2}\,. \qquad (3.3.64)$$

The effective interaction Lagrangian density $\mathcal{L}^{\text{int}}_{\text{eff}}$ is given by the following expression and there emerges the oriented ghost-ghost-gauge coupling in the internal loop,

$$\begin{aligned}
\mathcal{L}^{\text{int}}_{\text{eff}} &= \mathcal{L}_{\text{eff}} - \mathcal{L}^{\text{quad}}_{\text{eff}} \\
&= \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) + \bar{c}_\alpha(x)M^{\text{L}}_{\alpha,\beta}(A_{\gamma\mu}(x))c_\beta(x) \\
&\quad -\frac{1}{2}A^\mu_\alpha(x)\delta_{\alpha\beta}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu_\beta(x) - \bar{c}_\alpha(x)\delta_{\alpha\beta}(-\partial^2)c_\beta(x) \\
&= \frac{1}{2}C_{\alpha\beta\gamma}(\partial_\mu A_{\alpha\nu}(x) - \partial_\nu A_{\alpha\mu}(x))A^\mu_\beta(x)A^\nu_\gamma(x) \\
&\quad -\frac{1}{4}C_{\alpha\beta\gamma}C_{\alpha\delta\varepsilon}A_{\beta\mu}(x)A_{\gamma\nu}(x)A^\mu_\delta(x)A^\nu_\varepsilon(x) \\
&\quad +C_{\alpha\beta\gamma}\partial^\mu\bar{c}_\alpha(x)A_{\gamma\mu}(x)c_\beta(x)\,. \qquad (3.3.65)
\end{aligned}$$

**(3) Covariant Gauge**: We use the second Faddeev–Popov formula, (3.3.35a), (3.3.35b) and (3.3.35c). The kernel $K_{\alpha\mu,\beta\nu}(x-y;\xi)$ of the quadratic part of the effective action functional $I_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta]$ with respect to the gauge field $A_{\gamma\mu}(x)$ in the covariant gauge, (3.3.38),

$$\begin{aligned}
&I^{\text{quad. in } A^\mu_\gamma}_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta] \\
&= \int \mathrm{d}^4x\{\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) - \tfrac{1}{2}F^2_\alpha(A_{\gamma\mu}(x))\}^{\text{quad. in } A^\mu_\gamma} \\
&\equiv -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y A^\mu_\alpha(x)K_{\alpha\mu,\beta\nu}(x-y;\xi)A^\nu_\beta(y)\,, \qquad (3.3.66)
\end{aligned}$$

is given by

$$K_{\alpha\mu,\beta\nu}(x-y;\xi)=\delta_{\alpha\beta}\{-\eta_{\mu\nu}\partial^2+(1-\xi)\partial_\mu\partial_\nu\}\delta^4(x-y)\,, \tag{3.3.67}$$

and is invertible for $\xi>0$.

The "free" Green's function $D^{(\text{cov.})}_{\alpha\mu,\beta\nu}(x-y;\xi)$ of the gauge field $A_{\gamma\mu}(x)$ in the covariant gauge satisfies the following equation,

$$\delta_{\alpha\beta}\{\eta^{\mu\nu}\partial^2-(1-\xi)\partial^\mu\partial^\nu\}D^{(\text{cov.})}_{\beta\nu,\gamma\lambda}(x-y;\xi)=\delta_{\alpha\gamma}\eta^\mu_\lambda\delta^4(x-y)\,. \tag{3.3.68}$$

Upon Fourier transforming the "free" Green's function $D^{(\text{cov.})}_{\alpha\mu,\beta\nu}(x-y;\xi)$,

$$\begin{aligned}D^{(\text{cov.})}_{\alpha\mu,\beta\nu}(x-y;\xi)&=\delta_{\alpha\beta}D^{(\text{cov.})}_{\mu,\nu}(x-y;\xi)\\&=\delta_{\alpha\beta}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\exp[-\mathrm{i}k(x-y)]D^{(\text{cov.})}_{\mu,\nu}(k;\xi)\,,\end{aligned} \tag{3.3.69}$$

the momentum space Green's function $D^{(\text{cov.})}_{\mu,\nu}(k;\xi)$ satisfies

$$\{-\eta^{\mu\nu}k^2+(1-\xi)k^\mu k^\nu\}D^{(\text{cov.})}_{\nu,\lambda}(k;\xi)=\eta^\mu_\lambda\,. \tag{3.3.70a}$$

We define the four-dimensional transverse and longitudinal projection operators, $\Lambda_\mathrm{T}(k)$ and $\Lambda_\mathrm{L}(k)$, for an arbitrary 4-vector $k^\mu$ by

$$\Lambda_{\mathrm{T}\ \mu,\nu}(k)=\eta_{\mu,\nu}-\frac{k_\mu k_\nu}{k^2},\qquad \Lambda_{\mathrm{L}\ \mu,\nu}(k)=\frac{k_\mu k_\nu}{k^2}\,. \tag{3.3.71}$$

$$\Lambda^2_\mathrm{T}(k)=\Lambda_\mathrm{T}(k),\quad \Lambda^2_\mathrm{L}(k)=\Lambda_\mathrm{L}(k),\quad \Lambda_\mathrm{T}(k)\Lambda_\mathrm{L}(k)=\Lambda_\mathrm{L}(k)\Lambda_\mathrm{T}(k)=0\,. \tag{3.3.72}$$

Equation (3.3.70a) is written in terms of $\Lambda_\mathrm{T}(k)$ and $\Lambda_\mathrm{L}(k)$ as

$$(-k^2)(\Lambda^{\mu,\nu}_\mathrm{T}(k)+\xi\Lambda^{\mu,\nu}_\mathrm{L}(k))D^{(\text{cov.})}_{\nu,\lambda}(k;\xi)=\eta^\mu_\lambda\,. \tag{3.3.70b}$$

Expressing $D^{(\text{cov.})}_{\nu,\lambda}(k;\xi)$ as

$$D^{(\text{cov.})}_{\nu,\lambda}(k;\xi)=\Lambda_{\mathrm{T}\ \nu,\lambda}(k)A(k^2)+\Lambda_{\mathrm{L}\ \nu,\lambda}(k)B(k^2;\xi)\,, \tag{3.3.73}$$

and making use of (3.3.72), we obtain

$$(-k^2A(k^2))\Lambda^{\mu}_{\mathrm{T}\ \lambda}(k)+(-\xi k^2B(k^2;\xi))\Lambda^{\mu}_{\mathrm{L}\ \lambda}(k)=\eta^\mu_\lambda\,. \tag{3.3.74}$$

Left-multiplying $\Lambda_{\mathrm{T}\ \tau,\mu}(k)$ and $\Lambda_{\mathrm{L}\ \tau,\mu}(k)$ in (3.3.74), respectively, we obtain

$$\begin{aligned}&(-k^2A(k^2))\Lambda_{\mathrm{T}\ \tau,\lambda}(k)=\Lambda_{\mathrm{T}\ \tau,\lambda}(k),\\&(-\xi k^2B(k^2;\xi))\Lambda_{\mathrm{L}\ \tau,\lambda}(k)=\Lambda_{\mathrm{L}\ \tau,\lambda}(k)\,,\end{aligned} \tag{3.3.75}$$

from which we obtain

$$A(k^2) = -\frac{1}{k^2}, \qquad B(k^2;\xi) = -\frac{1}{\xi}\frac{1}{k^2}\,, \tag{3.3.76}$$

$$\begin{aligned} D^{(\text{cov.})}_{\nu,\lambda}(k;\xi) &= -\frac{1}{k^2}\left\{\left(\eta_{\nu,\lambda} - \frac{k_\nu k_\lambda}{k^2}\right) + \frac{1}{\xi}\frac{k_\nu k_\lambda}{k^2}\right\} \\ &= -\frac{1}{k^2}\left\{\eta_{\nu,\lambda} - \left(1 - \frac{1}{\xi}\right)\frac{k_\nu k_\lambda}{k^2}\right\}. \end{aligned} \tag{3.3.77}$$

The gauge parameter $\xi$ shifts the longitudinal component of the covariant gauge Green's function. In the limit $\xi \to \infty$, the covariant gauge Green's function $D^{(\text{cov.})}_{\nu,\lambda}(k;\xi)$ coincides with the Landau gauge Green's function $D^{(\text{Landau})}_{\nu,\lambda}(k)$, and, at $\xi = 1$, the covariant gauge Green's function $D^{(\text{cov.})}_{\nu,\lambda}(k;\xi)$ coincides with the $'$t Hooft–Feynman gauge Green's function $D^{(\text{Feynman})}_{\nu,\lambda}(k)$,

$$D^{(\text{cov.})}_{\nu,\lambda}(k;\infty) = -\frac{1}{k^2}\left(\eta_{\nu,\lambda} - \frac{k_\nu k_\lambda}{k^2}\right) = D^{(\text{Landau})}_{\nu,\lambda}(k)\,, \tag{3.3.78a}$$

$$D^{(\text{cov.})}_{\nu,\lambda}(k;1) = -\frac{1}{k^2}\eta_{\nu,\lambda} = D^{(\text{Feynman})}_{\nu,\lambda}(k)\,. \tag{3.3.78b}$$

We calculate the kernel $M^{(\text{cov.})}_{\alpha x,\beta y}(A_{\gamma\mu})$ of the Faddeev–Popov ghost in the covariant gauge.

$$\begin{aligned} M^{(\text{cov.})}_{\alpha x,\beta y}(A_{\gamma\mu}) &= \frac{\delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}(-D^{\text{adj}}_\mu)_{\gamma\beta}\delta^4(x-y) \\ &= \sqrt{\xi} M^{(\text{Landau})}_{\alpha x,\beta y}(A_{\gamma\mu}) \\ &= \sqrt{\xi} M^{\text{L}}_{\alpha,\beta}(A_{\gamma\mu}(x))\delta^4(x-y)\,. \end{aligned} \tag{3.3.79}$$

We absorb the factor $\sqrt{\xi}$ into the normalization of the Faddeev–Popov ghost field, $\bar{c}_\alpha(x)$ and $c_\beta(x)$. The ghost part of the effective action functional is then given by

$$\begin{aligned} I^{\text{ghost}}_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta] &= \int \mathrm{d}^4x \bar{c}_\alpha(x) M^{\text{L}}_{\alpha,\beta}(A_{\gamma\mu}(x))c_\beta(x) \\ &= \int \mathrm{d}^4x \partial^\mu \bar{c}_\alpha(x)(\partial_\mu c_\alpha(x) \\ &\quad + C_{\alpha\beta\gamma}A_{\gamma\mu}(x)c_\beta(x))\,, \end{aligned} \tag{3.3.80}$$

just like in the Landau gauge, (3.3.63). From this, we find that the Faddeev–Popov ghost in the covariant gauge is the massless scalar Fermion whose Green's function $D^{(C)}_{\mathrm{L}\ \alpha,\beta}(x-y)$ is given by

$$D^{(C)}_{\mathrm{L}\ \alpha,\beta}(x-y) = \delta_{\alpha\beta}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\exp[-\mathrm{i}k(x-y)]\frac{1}{k^2}\,. \tag{3.3.81}$$

We calculate the effective interaction Lagrangian density $\mathcal{L}^{\mathrm{int}}_{\mathrm{eff}}$ in the covariant gauge. We have

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} &= \mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(x)) - \frac{1}{2}F^2_{\alpha}(A_{\gamma\mu}(x))\\ &\quad+\bar{c}_{\alpha}(x)M^{\mathrm{L}}_{\alpha,\beta}(A_{\gamma\mu}(x))c_{\beta}(x)\,,\end{aligned} \tag{3.3.82}$$

and

$$\begin{aligned}\mathcal{L}^{\mathrm{quad}}_{\mathrm{eff}} &= -\frac{1}{2}A^{\mu}_{\alpha}(x)\delta_{\alpha\beta}\{-\eta_{\mu\nu}\partial^2+(1-\xi)\partial_{\mu}\partial_{\nu}\}A^{\nu}_{\beta}(x)\\ &\quad+\bar{c}_{\alpha}(x)\delta_{\alpha\beta}(-\partial^2)c_{\beta}(x)\,.\end{aligned} \tag{3.3.83}$$

From (3.3.82) and (3.3.83), we obtain $\mathcal{L}^{\mathrm{int}}_{\mathrm{eff}}$ as

$$\begin{aligned}\mathcal{L}^{\mathrm{int}}_{\mathrm{eff}} &= \mathcal{L}_{\mathrm{eff}} - \mathcal{L}^{\mathrm{quad}}_{\mathrm{eff}}\\ &= \frac{1}{2}C_{\alpha\beta\gamma}(\partial_{\mu}A_{\alpha\nu}(x)-\partial_{\nu}A_{\alpha\mu}(x))A^{\mu}_{\beta}(x)A^{\nu}_{\gamma}(x)\\ &\quad-\frac{1}{4}C_{\alpha\beta\gamma}C_{\alpha\delta\varepsilon}A_{\beta\mu}(x)A_{\gamma\nu}(x)A^{\mu}_{\delta}(x)A^{\nu}_{\varepsilon}(x)\\ &\quad+C_{\alpha\beta\gamma}\partial^{\mu}\bar{c}_{\alpha}A_{\gamma\mu}(x)c_{\beta}(x)\,.\end{aligned} \tag{3.3.84a}$$

The generating functional $Z_F[J_{\gamma\mu},\zeta_{\alpha},\bar{\zeta}_{\beta}]$ ($W_F[J_{\gamma\mu},\zeta_{\alpha},\bar{\zeta}_{\beta}]$) of (the connected parts of) the "full" Green's functions in the covariant gauge is given by

$$\begin{aligned}Z_F[J_{\gamma\mu},\zeta_{\alpha},\bar{\zeta}_{\beta}] &\equiv \exp[\mathrm{i}W_F[J_{\gamma\mu},\zeta_{\alpha},\bar{\zeta}_{\beta}]]\\ &\equiv \Big\langle 0,\mathrm{out}\Big|\mathrm{T}\Big(\exp\Big[\mathrm{i}\int\mathrm{d}^4z\{J_{\gamma\mu}(z)\hat{A}^{\mu}_{\gamma}(z)\\ &\qquad+\bar{c}_{\alpha}(z)\zeta_{\alpha}(z)+\bar{\zeta}_{\beta}(z)c_{\beta}(z)\}\Big]\Big)\Big|0,\mathrm{in}\Big\rangle\\ &= \int\mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\bar{c}_{\alpha}]\mathcal{D}[c_{\beta}]\\ &\qquad\times\exp\Big[\mathrm{i}\int\mathrm{d}^4z\{\mathcal{L}_{\mathrm{eff}}+J_{\gamma\mu}(z)A^{\mu}_{\gamma}(z)\\ &\qquad+\bar{c}_{\alpha}(z)\zeta_{\alpha}(z)+\bar{\zeta}_{\beta}(z)c_{\beta}(z)\}\Big]\end{aligned}$$

$$= \exp\left[ \mathrm{i} I_{\text{eff}}^{\text{int}} \left[ \frac{1}{\mathrm{i}} \frac{\delta}{\delta J_{\gamma}^{\mu}}, \mathrm{i} \frac{\delta}{\delta \zeta_{\alpha}}, \frac{1}{\mathrm{i}} \frac{\delta}{\delta \bar{\zeta}_{\beta}} \right] \right] Z_{F,0} \left[ J_{\gamma\mu}, \zeta_{\alpha}, \bar{\zeta}_{\beta} \right] , \tag{3.3.85}$$

with

$$I_{\text{eff}}^{\text{int}}[A_{\gamma\mu}, \bar{c}_{\alpha}, c_{\beta}] = \int \mathrm{d}^4 x \mathcal{L}_{\text{eff}}^{\text{int}}((3.3.84\text{a})) , \tag{3.3.84b}$$

and

$$\begin{aligned} &Z_{F,0}[J_{\gamma\mu}, \zeta_{\alpha}, \bar{\zeta}_{\beta}] \\ &\quad = \int \mathcal{D}[A_{\gamma\mu}] \mathcal{D}[\bar{c}_{\alpha}] \mathcal{D}[c_{\beta}] \exp\left[ \mathrm{i} \int \mathrm{d}^4 z \{ \mathcal{L}_{\text{eff}}^{\text{quad}}((3.3.83)) + J_{\gamma\mu}(z) A_{\gamma}^{\mu}(z) \right. \\ &\qquad \left. + \bar{c}_{\alpha}(z) \zeta_{\alpha}(z) + \bar{\zeta}_{\beta}(z) c_{\beta}(z) \} \right] \\ &\quad = \exp\left[ \mathrm{i} \int \mathrm{d}^4 x \mathrm{d}^4 y \{ -\tfrac{1}{2} J_{\alpha}^{\mu}(x) D_{\alpha\mu,\beta\nu}^{(\text{cov.})}(x-y;\xi) J_{\beta}^{\nu}(y) \right. \\ &\qquad \left. - \bar{\zeta}_{\beta}(x) D_{\alpha\beta}^{(C)}(x-y) \zeta_{\alpha}(y) \} \right] . \end{aligned} \tag{3.3.86}$$

Equation (3.3.85) is the starting point of the Feynman–Dyson expansion of the "full" Green's function in terms of the "free" Green's functions and the effective interaction vertices, (3.3.84a). Noting the fact that the Faddeev–Popov ghost appears only in the internal loops, we might as well set

$$\zeta_{\alpha}(z) = \bar{\zeta}_{\beta}(z) = 0,$$

in $Z_F[J_{\gamma\mu}, \zeta_{\alpha}, \bar{\zeta}_{\beta}]$ and define $Z_F[J_{\gamma\mu}]$ by

$$\begin{aligned} Z_F[J_{\gamma\mu}] &\equiv \exp[\mathrm{i} W_F[J_{\gamma\mu}]] \\ &\equiv \left\langle 0, \text{out} \left| \mathrm{T} \left( \exp\left[ \mathrm{i} \int \mathrm{d}^4 z J_{\gamma\mu}(z) \hat{A}_{\gamma}^{\mu}(z) \right] \right) \right| 0, \text{in} \right\rangle \\ &= \int \mathcal{D}[A_{\gamma\mu}] \mathcal{D}[\bar{c}_{\alpha}] \mathcal{D}[c_{\beta}] \exp\left[ \mathrm{i} \int \mathrm{d}^4 z \{ \mathcal{L}_{\text{eff}} + J_{\gamma\mu}(z) A_{\gamma}^{\mu}(z) \} \right] \\ &= \int \mathcal{D}[A_{\gamma\mu}] \Delta_F[A_{\gamma\mu}] \exp\left[ \mathrm{i} \int \mathrm{d}^4 z \{ \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(z)) \right. \\ &\qquad \left. - \tfrac{1}{2} F_{\alpha}^2(A_{\gamma\mu}(z)) + J_{\gamma\mu}(z) A_{\gamma}^{\mu}(z) \} \right] , \end{aligned} \tag{3.3.87}$$

which we shall use in the next section.

### 3.3.4 The Ward–Takahashi–Slavnov–Taylor Identity and Gauge Independence of the Physical $S$-Matrix

In the discussion so far, in order to accomplish path integral quantization of the gauge field, we introduced the gauge-fixing term and the requisite Faddeev–Popov ghost term in the original gauge-invariant Lagrangian density. In the discussion of the covariant gauge in Sect. 3.3.3, we saw the explicit $\xi$-dependence of the longitudinal part of the covariant gauge Green's function $D_{\mu\nu}^{(\mathrm{cov.})}(k;\xi)$. The physical S-matrix, however, should not depend on the choice of the gauge. The identity in the heading of this subsection is the starting point of the demonstration of the gauge independence of the physical $S$-matrix. This identity will be the last remnant of the gauge invariance of the original Lagrangian density.

In the functional integrand of $Z_F[J_{\gamma\mu}]$,

$$\begin{aligned} Z_F[J_{\gamma\mu}] = &\int \mathcal{D}[A_{\gamma\mu}]\varDelta_F[A_{\gamma\mu}] \\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(x)) - \tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}(z)) + J_{\gamma\mu}(z)A_\gamma^\mu(z)\}\right], \end{aligned} \tag{3.3.87}$$

we perform the nonlinear infinitesimal gauge transformation $g_0$, parametrized by

$$\varepsilon_\alpha(x;M^F(A_{\gamma\mu})) = (M^F(A_{\gamma\mu}))^{-1}_{\alpha x,\beta y}\lambda_\beta(y), \tag{3.3.27}$$

where $\lambda_\beta(y)$ is an arbitrary infinitesimal function independent of $A_{\gamma\mu}(x)$. Due to the invariance of the integration measure, $\mathcal{D}[A_{\gamma\mu}]\varDelta_F[A_{\gamma\mu}]$, and due to the fact that the value of the functional integral remains unchanged under the change of the function variable, we have the following identity,

$$\begin{aligned} 0 = &\int \mathcal{D}[A_{\gamma\mu}]\varDelta_F[A_{\gamma\mu}] \\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(z)) - \tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}(z)) + J_{\gamma\mu}(z)A_\gamma^\mu(z)\}\right] \\ &\times \frac{\delta}{\delta\lambda_\alpha(x)}\int \mathrm{d}^4 z\{-\tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}^{g_0}) + J_{\gamma\mu}(z)A_\gamma^{g_0\mu}(z)\}|_{g_0=1} \\ = &\int \mathcal{D}[A_{\gamma\mu}]\varDelta_F[A_{\gamma\mu}] \\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(z)) - \tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}(z)) + J_{\gamma\mu}(z)A_\gamma^\mu(z)\}\right] \\ &\times\{-F_\alpha(A_{\gamma\mu}(x)) + J_\varepsilon^\mu(x)(-D_\mu^{\mathrm{adj}}(A_{\gamma\mu}(x)))_{\varepsilon\beta}(M^F(A_{\gamma\mu}(x)))^{-1}_{\beta\alpha}\}. \end{aligned} \tag{3.3.88a}$$

Pulling out the last { } part in front of the functional integral, we obtain

$$\left\{-F_\alpha\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^\mu_\gamma(x)}\right)+J^\mu_\varepsilon(x)\left(-D^{\mathrm{adj}}_\mu\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^\mu_\gamma(x)}\right)\right)_{\varepsilon\beta}\right.$$
$$\left.\times\left(M^F\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^\mu_\gamma(x)}\right)\right)^{-1}_{\beta\alpha}\right\}Z_F[J_{\gamma\mu}]=0, \tag{3.3.88b}$$

where

$$\left(D^{\mathrm{adj}}_\mu\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^\mu_\gamma(x)}\right)\right)_{\alpha\beta}=\delta_{\alpha\beta}\partial_\mu+C_{\alpha\beta\gamma}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J^\mu_\gamma(x)}. \tag{3.3.89}$$

We call the identity, (3.3.88b), the Ward–Takahashi–Slavnov–Taylor identity.
With this identity, we shall prove the gauge independence of the physical $S$-matrix.

*Claim: The physical $S$-matrix is invariant under the infinitesimal variation $\Delta F_\alpha(A_{\gamma\mu}(x))$ of the gauge-fixing condition $F_\alpha(A_{\gamma\mu}(x))$.*

Under the variation of the gauge-fixing condition,

$$F_\alpha(A_{\gamma\mu}(x))\to F_\alpha(A_{\gamma\mu}(x))+\Delta F_\alpha(A_{\gamma\mu}(x)), \tag{3.3.90}$$

we have, to first order in $\Delta F_\alpha(A_{\gamma\mu}(x))$,

$$Z_{F+\Delta F}[J_{\gamma\mu}]-Z_F[J_{\gamma\mu}]=\int\mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]$$
$$\times\exp\left[\mathrm{i}\int\mathrm{d}^4z\{\mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(z))-\tfrac{1}{2}F^2_\alpha(A_{\gamma\mu}(z))+J^\mu_\gamma(z)A_{\gamma\mu}(z)\}\right]$$
$$\times\left\{-\mathrm{i}\int\mathrm{d}^4xF_\alpha(A_{\gamma\mu}(x))\Delta F_\alpha(A_{\gamma\mu}(x))+\frac{\Delta_{F+\Delta F}[A_{\gamma\mu}]-\Delta_F[A_{\gamma\mu}]}{\Delta_F[A_{\gamma\mu}]}\right\}. \tag{3.3.91}$$

We calculate $\Delta_{F+\Delta F}[A_{\gamma\mu}]$ to first order in $\Delta F_\alpha(A_{\gamma\mu}(x))$,

$$\begin{aligned}\Delta_{F+\Delta F}[A_{\gamma\mu}]&=\mathrm{Det}M^{F+\Delta F}_{\alpha x,\beta y}(A_{\gamma\mu})\\&=\mathrm{Det}\left\{\left(\frac{\delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}+\frac{\delta\Delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}\right)\right.\\&\qquad\left.\times(-D^{\mathrm{adj}}_\mu(A_{\varepsilon\mu}(x)))\delta^4(x-y)\right\}\\&=\mathrm{Det}M^F_{\alpha,\beta}(A_{\gamma\mu}(x))\mathrm{Det}\left\{\delta_{\alpha\beta}\delta^4(x-y)\right.\\&\qquad+\frac{\delta\Delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}(-D^{\mathrm{adj}}_\mu(A_{\varepsilon\mu}(x)))_{\gamma\zeta}\\&\qquad\left.\times(M^F(A_{\varepsilon\mu}(x)))^{-1}_{\zeta\beta}\delta^4(x-y)\right\}\end{aligned}$$

$$= \Delta_F[A_{\gamma\mu}]\Bigg\{1 + \int \mathrm{d}^4x \frac{\delta \Delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)} \times(-D_\mu^{\mathrm{adj}}(A_{\varepsilon\mu}(x)))_{\gamma\zeta}(M^F(A_{\varepsilon\mu}(x)))^{-1}_{\zeta\alpha} + O((\Delta F)^2)\Bigg\}.$$

From this, we obtain

$$\frac{\Delta_{F+\Delta F}[A_{\gamma\mu}] - \Delta_F[A_{\gamma\mu}]}{\Delta_F[A_{\gamma\mu}]} = \int \mathrm{d}^4x \frac{\delta \Delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)} \times(-D_\mu^{\mathrm{adj}}(A_{\varepsilon\mu}(x)))_{\gamma\zeta}(M^F(A_{\varepsilon\mu}(x)))^{-1}_{\zeta\alpha} + O((\Delta F)^2). \quad (3.3.92)$$

From (3.3.91) and (3.3.92), we obtain

$$\begin{aligned} Z_{F+\Delta F}[J_{\gamma\mu}] - Z_F[J_{\gamma\mu}] &= \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^4z\{\mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(z)) - \tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}(z)) + J_\gamma^\mu(z)A_{\gamma\mu}(z)\}\right] \\ &\times \Bigg\{-\mathrm{i}\int \mathrm{d}^4x F_\alpha(A_{\gamma\mu}(x))\Delta F_\alpha(A_{\gamma\mu}(x)) \\ &+ \frac{\delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}(-D_\mu^{\mathrm{adj}}(A_{\varepsilon\mu}(x)))_{\gamma\zeta}(M^F(A_{\varepsilon\mu}(x)))^{-1}_{\zeta\alpha} + O((\Delta F)^2)\Bigg\}. \end{aligned} \quad (3.3.93)$$

We now note the following identity,

$$\begin{aligned} &\mathrm{i}\Delta F_\alpha\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_\gamma^\mu(x)}\right) J_\gamma^\mu(x)\exp\left[\mathrm{i}\int \mathrm{d}^4z J_\gamma^\mu(z)A_{\gamma\mu}(z)\right] \\ &= \exp\left[\mathrm{i}\int \mathrm{d}^4z J_\gamma^\mu(z)A_{\gamma\mu}(z)\right] \\ &\quad \times \left\{\mathrm{i}J_\gamma^\mu(x)\Delta F_\alpha(A_{\gamma\mu}(x)) + \frac{\delta \Delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}\right\}. \end{aligned} \quad (3.3.94)$$

Left-multiplying by $\mathrm{i}\Delta F_\alpha\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_\gamma^\mu(x)}\right)$ in (3.3.88a), we obtain

$$
\begin{aligned}
0 = &\int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \\
&\times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(z)) - \tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}(z)) + J_\gamma^\mu(z)A_{\gamma\mu}(z)\}\right] \\
&\times \int \mathrm{d}^4 x\biggl\{ -\mathrm{i}F_\alpha(A_{\gamma\mu}(x))\Delta F_\alpha(A_{\gamma\mu}(x)) \\
&+\left(\mathrm{i}J_\gamma^\mu(x)\Delta F_\alpha(A_{\gamma\mu}(x)) + \frac{\delta \Delta F_\alpha(A_{\gamma\mu}(x))}{\delta A_{\gamma\mu}(x)}\right) \\
&\times(-D_\mu^{\text{adj}}(A_{\varepsilon\mu}(x)))_{\gamma\beta}(M^F(A_{\gamma\mu}(x)))^{-1}_{\beta\alpha}\biggr\}. \qquad (3.3.95)
\end{aligned}
$$

We subtract (3.3.95) from (3.3.93), and obtain

$$
\begin{aligned}
&Z_{F+\Delta F}[J_{\gamma\mu}] - Z_F[J_{\gamma\mu}] = \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \\
&\times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(z)) - \frac{1}{2}F_\alpha^2(A_{\gamma\mu}(z)) + J_\gamma^\mu(z)A_{\gamma\mu}(z)\}\right] \\
&\times\biggl\{\mathrm{i}\int \mathrm{d}^4 x J_\gamma^\mu(x)(-D_\mu^{\text{adj}}(A_{\varepsilon\mu}(x)))_{\gamma\beta}(M^F(A_{\varepsilon\mu}(x)))^{-1}_{\beta\alpha}\Delta F_\alpha(A_{\gamma\mu}(x)) \\
&+O((\Delta F)^2)\biggr\}.
\end{aligned}
$$

We exponentiate the above expression to first order in $\Delta F_\alpha(A_{\gamma\mu}(x))$ as

$$
\begin{aligned}
Z_{F+\Delta F}[J_{\gamma\mu}] = &\int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \qquad (3.3.96\text{a}) \\
&\times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(z)) - \tfrac{1}{2}F_\alpha^2(A_{\gamma\mu}(z)) + J_\gamma^\mu(z)A_{\gamma\mu}^{\tilde{g}_0^{-1}}(z)\}\right],
\end{aligned}
$$

where

$$
A_{\gamma\mu}^{\tilde{g}_0^{-1}}(z) = A_{\gamma\mu}(z) + (D_\mu^{\text{adj}}(A_{\varepsilon\mu}(z)))_{\gamma\beta}(M^F(A_{\varepsilon\mu}(z)))^{-1}_{\beta\alpha}\Delta F_\alpha(A_{\gamma\mu}(z)). \qquad (3.3.96\text{b})
$$

Here, $\tilde{g}_0$ is the gauge transformation parametrized by

$$
\varepsilon_\alpha(x; M^F(A_{\gamma\mu}), \Delta F_\beta(A_{\gamma\mu})) = (M^F(A_{\varepsilon\mu}(x)))^{-1}_{\alpha\beta}\cdot\Delta F_\beta(A_{\gamma\mu}(x)). \qquad (3.3.97)
$$

The response of the gauge-fixing condition

$$
F_\alpha(A_{\gamma\mu}(x))
$$

to this $\tilde{g}_0$ gives (3.3.90). In this manner, we find that the variation of the gauge-fixing condition, (3.3.90), gives rise to an additional vertex between the external hook $J_\gamma^\mu(z)$ and the gauge field $A_{\gamma\mu}(z)$,

$$J_{\gamma}^{\mu}(z)(A_{\gamma\mu}^{\tilde{g}_0^{-1}}(z) - A_{\gamma\mu}(z)). \tag{3.3.98}$$

We can take care of this extra vertex by the renormalization of the wave function renormalization constants of the external lines of the $S$-matrix on the mass shell. Namely, we have

$$S_{F+\Delta F} = \prod_e \left( \frac{Z_{F+\Delta F}}{Z_F} \right)_e^{1/2} \cdot S_F. \tag{3.3.99}$$

Here, $S_F$ and $Z_F$ are the $S$-matrix and the wave function renormalization constant in the gauge specified by $F_\alpha(A_{\gamma\mu}(x))$, and the subscript "e" indicates the external lines. From (3.3.99), we find that the renormalized $S$-matrix, $S_{\mathrm{ren}}$, given by

$$S_{\mathrm{ren}} \equiv S_F \Big/ \prod_e \left( Z_F^{1/2} \right)_e \tag{3.3.100}$$

is independent of $F_\alpha(A_{\gamma\mu}(x))$, i.e., is gauge independent.

## 3.4 Spontaneous Symmetry Breaking and the Gauge Field

In this section, we discuss the method with which we give a mass term to the gauge field without violating gauge invariance.

We consider the matter field Lagrangian density $\mathcal{L}_{\mathrm{matter}}(\hat{\phi}(x), \partial_\mu \hat{\phi}(x))$ which is globally $G$ invariant. When the field operator $\hat{\phi}(x)$ develops the non-zero vacuum expectation value in the direction of some generator $T_\alpha$ of the internal symmetry group $G$, we say that the internal symmetry $G$ is spontaneously broken by the ground state $|0, {}^{\mathrm{in}}_{\mathrm{out}}\rangle$. In this instance, there emerges a massless excitation, corresponding to each broken generator, and we call it the Nambu–Goldstone boson (Sect. 3.4.1). We call this phenomenon the Goldstone's theorem. The elimination of the Nambu–Goldstone boson becomes the main issue. We note that the global $G$ symmetry is exact, but spontaneously broken by the ground state $|0, {}^{\mathrm{in}}_{\mathrm{out}}\rangle$. Next, by invoking Weyl's gauge principle, we extend the global $G$ invariance of the matter system to the local $G$ invariance of the matter-gauge system,

$$\mathcal{L}_{\mathrm{matter}}(\hat{\phi}(x), D_\mu \hat{\phi}(x)) + \mathcal{L}_{\mathrm{gauge}}(F_{\gamma\mu\nu}(x)).$$

Under the appropriate local $G$ transformation (the local phase transformation of $\hat{\phi}(x)$ and the gauge transformation of $\hat{A}_{\gamma\mu}(x)$) which depends on the vacuum expectation value of $\hat{\phi}(x)$, the Nambu–Goldstone boson gets eliminated and emerges as the longitudinal mode of the gauge field corresponding to the broken generator and the said gauge field becomes a massive vector field with three degrees of freedom, two transverse modes and one longitudinal mode. The gauge field corresponding to the unbroken generator remains as the massless gauge field with two transverse modes only. Since the Nambu–Goldstone boson got eliminated and emerged as the longitudinal mode of the massive vector field, the total degrees of freedom of the matter-gauge system remain unchanged. We call this mass generating mechanism as the Higgs–Kibble mechanism (Sect. 3.4.2). We carry out the path integral quantization of this matter-gauge system in the $R_\xi$-gauge. The $R_\xi$-gauge is a gauge-fixing condition involving both the matter field and the gauge field linearly which eliminates the mixed term of the matter field and the gauge field in the quadratic part of the total effective Lagrangian density $\mathcal{L}_{\mathrm{tot}}^{\mathrm{quad}}$ and fixes the gauge (Sect. 3.4.3). We prove the gauge independence of the physical $S$-matrix with the use of the Ward–Takahashi–Slavnov–Taylor identity (Sect. 3.4.4).

### 3.4.1 Goldstone's Theorem

We designate the $n$-component real scalar field $\phi_{\mathrm{i}}(x)$, $i = 1, \ldots, n$, by vector notation,

$$\tilde{\phi}(x) = \begin{pmatrix} \phi_1(x) \\ \cdot \\ \cdot \\ \phi_n(x) \end{pmatrix}, \qquad \tilde{\phi}^{\mathrm{T}}(x) = \left( \phi_1(x), \ldots \ \phi_n(x) \right), \tag{3.4.1}$$

and express the globally $G$ invariant matter field Lagrangian density as

$$\mathcal{L}_{\mathrm{matter}}(\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x)) = \frac{1}{2} \partial_\mu \tilde{\phi}^{\mathrm{T}}(x) \partial^\mu \tilde{\phi}(x) - V(\tilde{\phi}(x)). \tag{3.4.2}$$

Here, we assume the following three conditions:

(1) $G$ is a semi-simple $N$-parameter Lie group whose $N$ generators we designate

$$T_\alpha, \qquad \alpha = 1, \ldots, N.$$

(2) Under the global $G$ transformation, $\tilde{\phi}(x)$ transform with the $n$-dimensional reducible representation, $\theta_\alpha$, $\alpha = 1, \ldots, N$,

$$\delta\phi_{\mathrm{i}}(x) = \mathrm{i}\varepsilon_\alpha (\theta_\alpha)_{ij} \phi_j(x), \qquad \alpha = 1, \ldots, N, \qquad i, j = 1, \ldots, n. \tag{3.4.3}$$

(3) $V(\tilde{\phi}(x))$ is the globally $G$ invariant quartic polynomial in $\tilde{\phi}(x)$ whose the global $G$ invariance condition is given by

$$\frac{\partial V(\tilde{\phi}(x))}{\partial \phi_{\mathrm{i}}(x)} (\theta_\alpha)_{ij} \phi_j(x) = 0, \qquad \alpha = 1, \ldots, N. \tag{3.4.4}$$

We assume that the minimizing $\tilde{\phi}(x)$ of $V(\tilde{\phi}(x))$ exists and is given by the constant vector $\tilde{v}$,

$$\frac{\partial V(\tilde{\phi}(x))}{\partial \phi_{\rm i}(x)}|_{\tilde{\phi}(x)=\tilde{v}} = 0, \qquad i = 1, \ldots, n. \tag{3.4.5}$$

We differentiate the global $G$ invariance condition, (3.4.4), with respect to $\phi_k(x)$ and set $\tilde{\phi}(x) = \tilde{v}$, obtaining the broken symmetry condition

$$\frac{\partial^2 V(\tilde{\phi}(x))}{\partial \phi_k(x)\partial \phi_{\rm i}(x)}|_{\tilde{\phi}(x)=\tilde{v}}(\theta_\alpha)_{ij}v_j = 0, \quad \alpha = 1, \ldots, N, \quad i, j, k = 1, \ldots, n. \tag{3.4.6}$$

We expand $V(\tilde{\phi}(x))$ around $\tilde{\phi}(x) = \tilde{v}$ and choose $V(\tilde{v}) = 0$ as the origin of energy.

$$V(\tilde{\phi}(x)) = \frac{1}{2}(\tilde{\phi}(x) - \tilde{v})^{\rm T}(M^2)(\tilde{\phi}(x) - v) + \text{higher-order terms.} \tag{3.4.7a}$$

We set

$$(M^2)_{i,j} \equiv \frac{\partial^2 V(\tilde{\phi}(x))}{\partial \phi_{\rm i}(x)\partial \phi_j(x)}|_{\tilde{\phi}(x)=\tilde{v}} \qquad i, j = 1, \ldots, n, \tag{3.4.7b}$$

where $M^2$ is the mass matrix of the $n$-component real scalar field. We express the broken symmetry condition, (3.4.6), in terms of the mass matrix $M^2$ as

$$(M^2)_{i,j}(\theta_\alpha \tilde{v})_j = 0, \qquad \alpha = 1, \ldots, N, \qquad i, j = 1, \ldots, n. \tag{3.4.8}$$

We let the stability group of the vacuum, i.e., the symmetry group of the ground state $\tilde{\phi}(x) = \tilde{v}$, be the $M$-dimensional subgroup $S \subset G$. When $\theta_\alpha$ is the realization on the scalar field $\tilde{\phi}(x)$ of the generator $T_\alpha$ belonging to the stability group $S$, this $\theta_\alpha$ annihilates the "vacuum" $\tilde{v}$

$$(\theta_\alpha \tilde{v}) = 0 \quad \text{for} \quad \theta_\alpha \in S, \tag{3.4.9}$$

and we have the invariance of $\tilde{v}$ expressed as

$$\exp\left[{\rm i}\sum_{\theta_\alpha \in S} \varepsilon_\alpha \theta_\alpha\right]\tilde{v} = \tilde{v}, \quad \text{stability group of the vacuum.} \tag{3.4.10}$$

As for the $M$ realizations $\theta_\alpha \in S$ on the scalar field $\tilde{\phi}(x)$ of the $M$ generators $T_\alpha \in S$, the broken symmetry condition, (3.4.8), is satisfied automatically due to the stability condition, (3.4.9), and we do not get any new information from (3.4.8). As for the remaining $(N - M)$ realizations $\theta_\alpha \notin S$ on the scalar field $\tilde{\phi}(x)$ of the $(N - M)$ broken generators $T_\alpha \notin S$ which break the stability condition, (3.4.9), we get the following information from (3.4.8), i.e., the "mass matrix" $(M^2)_{i,j}$ has $(N-M)$ non-trivial eigenvectors belonging to the eigenvalue 0,

$$\theta_\alpha \tilde{v} \neq 0, \qquad \theta_\alpha \notin S. \tag{3.4.11}$$

Here, we show that $\{\theta_\alpha \tilde{v}, \theta_\alpha \notin S\}$ span the $(N-M)$-dimensional vector space. We define the $N \times N$ matrix $\mu^2_{\alpha,\beta}$ by

$$\mu^2_{\alpha,\beta} \equiv (\theta_\alpha \tilde{v}, \theta_\beta \tilde{v}) \equiv \sum_{i=1}^{n} (\theta_\alpha \tilde{v})^\dagger_{\rm i} (\theta_\beta \tilde{v})_i, \quad \alpha, \beta = 1, \ldots, N. \tag{3.4.12a}$$

From the Hermiticity of $\theta_\alpha$, we have

$$\mu^2_{\alpha,\beta} = (\tilde{v}, \theta_\alpha \theta_\beta \tilde{v}) = \sum_{i=1}^{n} v_i^* (\theta_\alpha \theta_\beta \tilde{v})_i, \tag{3.4.12b}$$

$$\mu^2_{\alpha,\beta} - \mu^2_{\beta,\alpha} = (\tilde{v}, [\theta_\alpha, \theta_\beta] \tilde{v}) = \mathrm{i} C_{\alpha\beta\gamma} (\tilde{v}, \theta_\gamma \tilde{v}) = 0, \tag{3.4.13}$$

i.e., we know that $\mu^2_{\alpha,\beta}$ is a real symmetric matrix. The last equality of (3.4.13) follows from the antisymmetry of $\theta_\alpha$, (3.2.67). Next, we let $\tilde{\mu}^2_{\alpha,\beta}$ be the restriction of $\mu^2_{\alpha,\beta}$ to the subspace $\{\theta_\alpha \tilde{v}, \theta_\alpha \notin S\}$. The $\tilde{\mu}^2_{\alpha,\beta}$ is a real symmetric $(N-M) \times (N-M)$ matrix and is diagonalizable. We let the $(N-M) \times (N-M)$ orthogonal matrix which diagonalizes $\tilde{\mu}^2_{\alpha,\beta}$ be $O$, and let $\tilde{\mu}'^2_{\alpha,\beta}$ be the diagonalized $\tilde{\mu}^2_{\alpha,\beta}$:

$$\tilde{\mu}'^2_{\alpha,\beta} = (O \tilde{\mu}^2 O)_{\alpha,\beta} = ((O\theta)_\alpha \tilde{v}, (O\theta)_\beta \tilde{v}) = \delta_{\alpha\beta} \cdot \tilde{\mu}'^2_{(\alpha)}. \tag{3.4.14}$$

The diagonal element $\tilde{\mu}'^2_{(\alpha)}$ is given by

$$\tilde{\mu}'^2_{(\alpha)} = ((O\theta)_\alpha \tilde{v}, (O\theta)_\alpha \tilde{v}) = \sum_{i=1}^{n} ((O\theta)_\alpha \tilde{v})^\dagger_i ((O\theta)_\alpha \tilde{v})_i. \tag{3.4.15}$$

From the definition of the stability group of the vacuum, (3.4.9), and the definition of linear independence in the $N$-dimensional vector space spanned by the realization $\theta_\alpha$, we obtain the following three statements:

(1) $\forall \theta_\alpha \notin S : (O\theta)_\alpha \tilde{v} \neq 0$.
(2) The $(N-M)$ diagonal elements, $\tilde{\mu}'^2_{(\alpha)}$, of $\tilde{\mu}'^2_{\alpha,\beta}$ are all positive.
(3) The $(N-M)$ $\theta'_\alpha$s, $\theta_\alpha \notin S$, are linearly independent.

Hence, we understand that $\{\theta_\alpha \tilde{v} : \forall \theta_\alpha \notin S\}$ span the $(N-M)$-dimensional vector space. In this way, we obtain the following theorem.

**Goldstone's Theorem**: When the global symmetry induced by $(N-M)$ generators $T_\alpha$ corresponding to $\{\theta_\alpha : \theta_\alpha \notin S\}$ is spontaneously broken $(\theta_\alpha \tilde{v} \neq$

0) by the vacuum $\tilde{v}$, the "mass matrix" $(M^2)_{i,j}$ has $(N-M)$ eigenvectors $\{\theta_\alpha \tilde{v} : \theta_\alpha \notin S\}$ belonging to the eigenvalue 0, and these vectors $\{\theta_\alpha \tilde{v} : \theta_\alpha \notin S\}$ span the $(N-M)$-dimensional vector space of the Nambu–Goldstone boson (massless excitation). The remaining $n-(N-M)$ scalar fields are massive.

From the preceding argument, we find that the $N \times N$ matrix $\mu^2_{\alpha,\beta}$ has rank $(N-M)$ and has $(N-M)$ positive eigenvalues $\tilde{\mu}'^2_{(\alpha)}$ and the $M$ zero eigenvalues. Now, we rearrange the generators $T_\alpha$ in the order of the $M$ unbroken generators $T_\alpha \in S$ and the $(N-M)$ broken generators $T_\alpha \notin S$. We get $\mu^2_{\alpha,\beta}$ in block diagonal form,

$$\mu^2_{\alpha,\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\mu}^2_{\alpha,\beta} \end{pmatrix}, \tag{3.4.16}$$

where the upper left diagonal block corresponds to the $M$-dimensional vector space of the stability group of the vacuum, $S$, and the lower right diagonal block corresponds to the $(N-M)$-dimensional vector space of the Nambu–Goldstone boson.

The purpose of the introduction of the $N \times N$ matrix $\mu^2_{\alpha,\beta}$ is twofold.

(1) $\{\theta_\alpha \tilde{v} : \theta_\alpha \notin S\}$ span the $(N-M)$-dimensional vector space.
(2) The "mass matrix" of the gauge fields to be introduced by the application of Weyl's gauge principle in the presence of the spontaneous symmetry breaking is given by $\mu^2_{\alpha,\beta}$.

Having disposed of the first point, we move on to a discussion of the second point.

### 3.4.2 Higgs–Kibble Mechanism

In this section, we extend the global $G$ invariance which is spontaneously broken by the "vacuum" $\tilde{v}$ of the matter field Lagrangian density

$$\mathcal{L}_{\text{matter}}(\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x))$$

to local $G$ invariance with Weyl's gauge principle. From the discussion of Sect. 3.2, we know that the total Lagrangian density $\mathcal{L}_{\text{total}}$ of the matter-gauge system after the gauge extension is given by

$$\begin{aligned}
&\mathcal{L}_{\text{total}} \\
&\quad = \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) + \mathcal{L}_{\text{scalar}}(\tilde{\phi}(x), D_\mu \tilde{\phi}(x)) \\
&\quad = -\frac{1}{4} F_{\gamma\mu\nu}(x) F^{\mu\nu}_\gamma(x) \\
&\qquad + \frac{1}{2}\{(\partial_\mu + \mathrm{i}\theta_\alpha A_{\alpha\mu}(x))\tilde{\phi}(x))\}^{\mathrm{T}}_i \{(\partial^\mu + \mathrm{i}\theta_\beta A^\mu_\beta(x))\tilde{\phi}(x)\}_i - V(\tilde{\phi}(x))
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}F_{\gamma\mu\nu}(x)F_{\gamma}^{\mu\nu}(x) \\
&+\frac{1}{2}\{\partial_\mu\tilde{\phi}^{\mathrm{T}}(x) - \mathrm{i}\tilde{\phi}^{\mathrm{T}}(x)\theta_\alpha A_{\alpha\mu}(x)\}_i\{\partial^\mu\tilde{\phi}(x) \\
&+\mathrm{i}\theta_\beta A_\beta^\mu(x)\tilde{\phi}(x)\}_i - V(\tilde{\phi}(x)),
\end{aligned} \tag{3.4.17}
$$

where, in the transition from the second equality to the third equality, we used the antisymmetry of $\theta_\alpha$, (3.2.67). We parametrize the $n$-component scalar field $\tilde{\phi}(x)$ as

$$
\tilde{\phi}(x) = \exp\left[\mathrm{i}\sum_{\theta_\alpha \notin S}\frac{\xi_\alpha(x)\theta_\alpha}{v_\alpha}\right](\tilde{v} + \tilde{\eta}(x)). \tag{3.4.18}
$$

We note that the $(N-M)$ $\xi_\alpha(x)'$s correspond to the $(N-M)$ broken generators $T_\alpha \notin S$, and the $\tilde{\eta}(x)$ has $(n-(N-M))$ non-vanishing components and is orthogonal to the Nambu–Goldstone direction, $\{\theta_\alpha\tilde{v} : \theta_\alpha \notin S\}$. We have the vacuum expectation values of $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ and $\{\eta_i(x)\}_{i=1}^{n-(N-M)}$ equal to 0. From the global $G$ invariance of $V(\tilde{\phi}(x))$, we have

$$
V(\tilde{\phi}(x)) = V(\tilde{v} + \tilde{\eta}(x)), \tag{3.4.19}
$$

and find that $V(\tilde{\phi}(x))$ is independent of $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$. Without the gauge extension, we had the $\xi_\alpha(x)$-dependence from the derivative term in $\mathcal{L}_{\mathrm{scalar}}$,

$$
\frac{1}{2}\partial_\mu\tilde{\phi}^{\mathrm{T}}(x)\partial^\mu\tilde{\phi}(x) = \frac{1}{2}\partial_\mu\xi_\alpha(x)\partial^\mu\xi_\alpha(x) + \cdots . \tag{3.4.20}
$$

Thus, the $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ would have been the massless Nambu–Goldstone boson with the gradient coupling with the other fields. As in (3.4.17), after the gauge extension, we have the freedom of the gauge transformation and can eliminate the Nambu–Goldstone boson fields $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ completely. We employ the following local phase transformation and the nonlinear gauge transformation:

$$
\begin{aligned}
\tilde{\phi}'(x) &= \exp\left[-\mathrm{i}\sum_{\theta_\alpha \notin S}\frac{\xi_\alpha(x)\theta_\alpha}{v_\alpha}\right]\tilde{\phi}(x) \\
&= \tilde{v} + \tilde{\eta}(x) \\
&= \begin{pmatrix} v_1 \\ \cdot \\ v_{N-M} \\ \eta_1(x) \\ \cdot \\ \eta_{n-(N-M)}(x) \end{pmatrix},
\end{aligned} \tag{3.4.21a}
$$

$$\theta_\gamma A'_{\gamma\mu}(x) = \exp\left[-\mathrm{i}\sum_{\theta_\alpha \notin S}\frac{\xi_\alpha(x)\theta_\alpha}{v_\alpha}\right]\left\{\theta_\gamma A_{\gamma\mu}(x) + \exp\left[\mathrm{i}\sum_{\theta_\beta \notin S}\frac{\xi_\beta(x)\theta_\beta}{v_\beta}\right]\left(\mathrm{i}\partial_\mu \exp\left[-\mathrm{i}\sum_{\theta_\beta \notin S}\frac{\xi_\beta(x)\theta_\beta}{v_\beta}\right]\right)\right\} \times \exp\left[\mathrm{i}\sum_{\theta_\alpha \notin S}\frac{\xi_\alpha(x)\theta_\alpha}{v_\alpha}\right]. \tag{3.4.21b}$$

As a result of these local $G$ transformations, we have managed to eliminate $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ completely and have the total Lagrangian density $\mathcal{L}_{\text{total}}$ as

$$\begin{aligned}\mathcal{L}_{\text{total}} = &-\frac{1}{4}F'_{\gamma\mu\nu}(x)F'^{\mu\nu}_{\gamma}(x)\\ &+\frac{1}{2}\{(\partial_\mu + \mathrm{i}\theta_\alpha A'_{\alpha\mu}(x))(\tilde{v}+\tilde{\eta}(x))\}^{\mathrm{T}}\\ &\times\{(\partial^\mu + \mathrm{i}\theta_\alpha A'^{\mu}_{\alpha}(x))(\tilde{v}+\tilde{\eta}(x))\} - V(\tilde{v}+\tilde{\eta}(x)).\end{aligned} \tag{3.4.22}$$

After the gauge transformations, (3.4.21a) and (3.4.21b), we have the covariant derivative as,

$$(\partial_\mu + \mathrm{i}\theta_\alpha A'_{\alpha\mu}(x))(\tilde{v}+\tilde{\eta}(x)) = \partial_\mu\eta(x) + \mathrm{i}(\theta_\alpha\tilde{v})A'_{\alpha\mu}(x) + \mathrm{i}(\theta_\alpha\tilde{\eta}(x))A'_{\alpha\mu}(x), \tag{3.4.23}$$

so that from the definitions of $\mu^2_{\alpha,\beta}$, (3.4.12a) and (3.4.12b), we have

$$\begin{aligned}&\frac{1}{2}\{(\partial_\mu + \mathrm{i}\theta_\alpha A'_{\alpha\mu}(x))(\tilde{v}+\tilde{\eta}(x))\}^{\mathrm{T}}\{(\partial^\mu + \mathrm{i}\theta_\beta A'^{\mu}_{\beta}(x))(\tilde{v}+\tilde{\eta}(x))\}\\ &= \frac{1}{2}(\partial_\mu\tilde{\eta}^{\mathrm{T}}(x)\partial^\mu\tilde{\eta}(x) + \mu^2_{\alpha,\beta}A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x)\\ &\quad+(\tilde{\eta}(x),\theta_\alpha\theta_\beta\tilde{\eta}(x))A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x)) + +\partial_\mu\tilde{\eta}^{\mathrm{T}}(x)\mathrm{i}(\theta_\alpha\tilde{v})A'^{\mu}_{\alpha}(x)\\ &\quad+\partial_\mu\tilde{\eta}^{\mathrm{T}}(x)\mathrm{i}(\theta_\alpha\tilde{\eta}(x))A'^{\mu}_{\alpha}(x) + (\tilde{\eta}(x),\theta_\alpha\theta_\beta\tilde{v})A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x).\end{aligned} \tag{3.4.24}$$

We expand $V(\tilde{\phi}'(x))$ around $\tilde{\phi}'(x) = \tilde{v}$, and obtain

$$\begin{aligned}V(\tilde{v}+\tilde{\eta}(x)) = &\frac{1}{2}(M^2)_{ij}\eta_i(x)\eta_j(x) + \frac{1}{3!}f_{ijk}\eta_i(x)\eta_j(x)\eta_k(x)\\ &+\frac{1}{4!}f_{ijkl}\eta_i(x)\eta_j(x)\eta_k(x)\eta_l(x),\end{aligned} \tag{3.4.25a}$$

$$(M^2)_{ij} = \frac{\partial^2 V(\tilde{\phi}(x))}{\partial\phi_i(x)\partial\phi_j(x)}|_{\tilde{\phi}(x)=\tilde{v}}, \tag{3.4.25b}$$

$$f_{ijk} = \frac{\partial^3 V(\tilde{\phi}(x))}{\partial\phi_i(x)\partial\phi_j(x)\partial\phi_k(x)}|_{\tilde{\phi}(x)=\tilde{v}}, \tag{3.4.25c}$$

$$f_{ijkl} = \frac{\partial^4 V(\tilde{\phi}(x))}{\partial\phi_i(x)\partial\phi_j(x)\partial\phi_k(x)\partial\phi_l(x)}|_{\tilde{\phi}(x)=\tilde{v}}. \tag{3.4.25d}$$

From (3.4.24) and (3.4.25a), we have $\mathcal{L}_{\text{total}}$ after the local $G$ transformation as

$$\begin{aligned}\mathcal{L}_{\text{total}} = & -\frac{1}{4}F'_{\gamma\mu\nu}(x)F'^{\mu\nu}_{\gamma}(x) + \frac{1}{2}\mu^2_{\alpha,\beta}A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x) + \partial_\mu\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{v})A'^{\mu}_{\alpha}(x) \\ & +\frac{1}{2}\partial_\mu\tilde{\eta}(x)\partial^\mu\tilde{\eta}(x) - \frac{1}{2}(M^2)_{i,j}\eta_i(x)\eta_j(x) + \partial_\mu\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{\eta}(x))A'^{\mu}_{\alpha}(x) \\ & +\tilde{\eta}(x)(\theta_\alpha\theta_\beta\tilde{v})A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x) - \frac{1}{3!}f_{ijk}\eta_i(x)\eta_j(x)\eta_k(x) \\ & +\frac{1}{2}\tilde{\eta}(x)(\theta_\alpha\theta_\beta\tilde{\eta}(x))A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x) - \frac{1}{4!}f_{ijkl}\eta_i(x)\eta_j(x)\eta_k(x)\eta_l(x).\end{aligned} \tag{3.4.26}$$

From (3.4.26), we obtain as the quadratic part of $\mathcal{L}_{\text{total}}$,

$$\begin{aligned}\mathcal{L}^{\text{quad}}_{\text{total}} = & -\frac{1}{4}(\partial_\mu A'_{\alpha\nu}(x) - \partial_\nu A'_{\alpha\mu}(x))(\partial^\mu A'^{\nu}_{\alpha}(x) - \partial^\nu A'^{\mu}_{\alpha}(x)) \\ & +\frac{1}{2}\mu^2_{\alpha,\beta}A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x) \\ & +\frac{1}{2}\partial_\mu\tilde{\eta}(x)\partial^\mu\tilde{\eta}(x) - \frac{1}{2}(M^2)_{i,j}\eta_i(x)\eta_j(x) \\ & +\partial_\mu\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{v})A'^{\mu}_{\alpha}(x).\end{aligned} \tag{3.4.27}$$

Were it not for the last term in (3.4.27) which represents the mixing of the gauge field and the scalar field, we can regard (3.4.27) as the "free" Lagrangian density of the following fields:

(1) $M$ massless gauge fields corresponding to the $M$ unbroken generators, $T_\alpha \in S$, belonging to the stability group of the vacuum,
(2) $(N - M)$ massive vector fields corresponding to the $(N - M)$ broken generators, $T_\alpha \notin S$, with the mass eigenvalues, $\tilde{\mu}'^2_{(\alpha)}$, $\alpha = M + 1, \ldots, N$,
(3) $(n - (N - M))$ massive scalar fields with the mass matrix, $(M^2)_{i,j}$.

The $(N-M)$ Nambu–Goldstone boson fields $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ get completely eliminated from the particle spectrum as a result of the gauge transformations, (3.4.21a) and (3.4.21b), and are absorbed as the longitudinal mode of the gauge fields corresponding to the $(N-M)$ broken generators $T_\alpha \notin S$. The said $(N-M)$ gauge fields become the $(N-M)$ massive vector fields with two transverse modes and one longitudinal mode. We call this mass generating mechanism for the gauge fields the *Higgs–Kibble mechanism*. We make lists of the degrees of freedom of the matter-gauge system before and after the gauge transformations, (3.4.21a) and (3.4.21b):

| Before the gauge transformation | Degrees of freedom |
|---|---|
| $N$ massless gauge fields | $2N$ |
| $(N-M)$ Goldstone boson fields | $N-M$ |
| $(n-(N-M))$ massive scalar fields | $n-(N-M)$ |
| Total degrees of freedom | $n+2N$ |

and

| After the gauge transformation | Degrees of freedom |
|---|---|
| $M$ massless gauge fields | $2M$ |
| $(N-M)$ massive vector fields | $3(N-M)$ |
| $(n-(N-M))$ massive scalar fields | $n-(N-M)$ |
| Total degrees of freedom | $n+2N$ |

There is no change in the total degrees of freedom, $n+2N$. Before the gauge transformation, the local $G$ invariance of $\mathcal{L}_{\text{total}}$, (3.4.17), is manifest, whereas after the gauge transformation, the particle spectrum content of $\mathcal{L}_{\text{total}}$, (3.4.26), is manifest and the local $G$ invariance of $\mathcal{L}_{\text{total}}$ is hidden. In this way, we can give the mass term to the gauge field without violating the local $G$ invariance and verify that the mass matrix of the gauge field is indeed given by $\mu^2_{\alpha,\beta}$.

### 3.4.3 Path Integral Quantization of the Gauge Field in the $R_\xi$-Gauge

In this section, we employ the $R_\xi$-gauge as the gauge-fixing condition, carry out the path integral quantization of the matter-gauge system described by the total Lagrangian density, (3.4.26), and, at the same time, eliminate the mixed term

$$\partial_\mu \tilde{\eta}(x) \mathrm{i}(\theta_\alpha \tilde{v}) A'^{\mu}_{\alpha}(x)$$

of the scalar fields $\tilde{\eta}(x)$ and the gauge fields $A'_{\alpha\mu}(x)$ in $\mathcal{L}^{\text{quad}}_{\text{total}}$. The $R_\xi$-gauge is the gauge-fixing condition involving the Higgs scalar field and the gauge field, $\{\tilde{\eta}(x), A'_{\gamma\mu}(x)\}$, linearly. We will use the general notation introduced in Appendix 4:

$$\{\phi_a\} = \{\eta_i(x),\ A'_{\gamma\mu}(x)\}, \qquad a = (\mathrm{i}, x),\ (\gamma\mu, x). \tag{3.4.28}$$

As the general linear gauge-fixing condition, we employ

$$F_\alpha(\{\phi_a\}) = a_\alpha(x), \qquad \alpha = 1, \dots, N. \tag{3.4.29}$$

We assume that

$$F_\alpha(\{\phi_a^g\}) = a_\alpha(x), \qquad \alpha = 1, \dots, N, \qquad \{\phi_a\} \text{ fixed}, \tag{3.4.30}$$

has the unique solution $g(x) \in G$. We parameterize the element $g(x)$ in the neighborhood of the identity element of $G$ by

$$g(x) = 1 + \mathrm{i}\varepsilon_\alpha(x)\theta_\alpha + O(\varepsilon^2), \tag{3.4.31}$$

with $\varepsilon_\alpha(x)$ an arbitrary infinitesimal function independent of $\{\phi_a\}$. The Faddeev–Popov determinant $\Delta_F[\{\phi_a\}]$ of the gauge-fixing condition, (3.4.29), is defined by

$$\Delta_F[\{\phi_a\}] \int \prod_x \mathrm{d}g(x) \prod_{\alpha,x} \delta(F_\alpha(\{\phi_a^g\}) - a_\alpha(x)) = 1, \tag{3.4.32}$$

and is invariant under the linear gauge transformation, (3.4.31),

$$\Delta_F[\{\phi_a^g\}] = \Delta_F\left[\{\phi_a\}\right]. \tag{3.4.33}$$

According to the first Faddeev–Popov formula, (3.24a), the vacuum-to-vacuum-transition amplitude of the matter-gauge system described by the total Lagrangian density, $\mathcal{L}_{\text{total}}((3.4.26))$, is given by

$$\begin{aligned} \mathcal{Z}_F(a_\alpha(x)) \equiv \langle 0, \text{out}|0, \text{in}\rangle_F = & \int \mathcal{D}[\{\phi_a\}]\Delta_F[\{\phi_a\}] \exp[\mathrm{i}I_{\text{total}}[\{\phi_a\}]] \\ & \times \prod_{\alpha,x} \delta(F_\alpha(\{\phi_a\}) - a_\alpha(x)), \end{aligned} \tag{3.4.34}$$

with

$$\mathcal{D}[\{\phi_a\}] \equiv \mathcal{D}[\eta_i(x)]\mathcal{D}\left[A'_{\gamma\mu}(x)\right], \tag{3.4.35}$$

and

$$I_{\text{total}}[\{\phi_a\}] = \int \mathrm{d}^4x \mathcal{L}_{\text{total}}((3.4.26)). \tag{3.4.36}$$

Since $\Delta_F[\{\phi_a\}]$ in $\langle 0, \text{out}|0,\text{in}\rangle_F$ gets multiplied by $\delta(F_\alpha(\{\phi_a\}) - a_\alpha(x))$, it is sufficient to calculate $\Delta_F[\{\phi_a\}]$ for $\{\phi_a\}$ which satisfies (3.4.29). Parametrizing $g(x)$ as in (3.4.31), we have

$$\Delta_F[\{\phi_a\}] = \mathrm{Det} M_F(\{\phi_a\}), \tag{3.4.37}$$

$$\{M_F(\{\phi_a\})\}_{\alpha x,\beta y} = \frac{\delta F_\alpha(\{\phi_a^g(x)\})}{\delta \varepsilon_\beta(y)}|_{g=1}, \quad \alpha, \beta = 1, \ldots, N, \tag{3.4.38}$$

$$F_\alpha(\{\phi_a^g\}) = F_\alpha(\{\phi_a\}) + M_F(\{\phi_a\})_{\alpha x,\beta y}\varepsilon_\beta(y) + O(\varepsilon^2). \tag{3.4.39}$$

We now consider the nonlinear gauge transformation $g_0$ parametrized by

$$\varepsilon_\alpha(x; \{\phi_a\}) = \{M_F^{-1}(\{\phi_a\})\}_{\alpha x,\beta y}\lambda_\beta(y), \tag{3.4.40}$$

with $\lambda_\beta(y)$ an arbitrary infinitesimal function independent of $\{\phi_a\}$. Under this nonlinear gauge transformation, we have

(1) $I_{\mathrm{total}}[\{\phi_a\}]$ is gauge invariant,

(2) $\mathcal{D}[\{\phi_a\}]\Delta_F[\{\phi_a\}] =$ gauge invariant measure, (3.4.41)

(for the proof, see Appendix 4);

(3) the gauge-fixing condition $F_\alpha(\{\phi_a\})$ gets transformed into

$$F_\alpha(\{\phi_a^{g_0}\}) = F_\alpha(\{\phi_a\}) + \lambda_\alpha(x) + O(\lambda^2). \tag{3.4.42}$$

Since the value of the functional integral remains unchanged under a change of function variables, the value of $\mathcal{Z}_F(a_\alpha(x))$ remains unchanged. Choosing

$$\lambda_\alpha(x) = \delta a_\alpha(x), \qquad \alpha = 1, \ldots, N, \tag{3.4.43}$$

we have

$$\mathcal{Z}_F(a_\alpha(x)) = \mathcal{Z}_F(a_\alpha(x) + \delta a_\alpha(x)) \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}a_\alpha(x)}\mathcal{Z}_F(a_\alpha(x)) = 0. \tag{3.4.44}$$

Since we find that $\mathcal{Z}_F(a_\alpha(x))$ is independent of $a_\alpha(x)$, we introduce an arbitrary weighting functional $H[a_\alpha(x)]$ for $\mathcal{Z}_F(a_\alpha(x))$ and path-integrate with respect to $a_\alpha(x)$, obtaining as the weighted $\mathcal{Z}_F(a_\alpha(x))$,

$$\begin{aligned}\mathcal{Z}_F &\equiv \int \prod_{\alpha,x} da_\alpha(x) H[a_\alpha(x)] \mathcal{Z}_F(a_\alpha(x)) \\ &= \int \mathcal{D}[\{\phi_a\}]\Delta_F[\{\phi_a\}] H[F_\alpha(\{\phi_a\})] \exp\left[\mathrm{i} I_{\mathrm{total}}[\{\phi_a\}]\right].\end{aligned} \tag{3.4.45}$$

As the weighting functional $H[a_\alpha(x)]$, we choose the quasi-Gaussian functional

$$H[a_\alpha(x)] = \exp\left[-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x a_\alpha^2(x)\right], \tag{3.4.46}$$

and obtain $\mathcal{Z}_F$ as

$$\begin{aligned}\mathcal{Z}_F &= \int \mathcal{D}[\{\phi_a\}]\Delta_F[\{\phi_a\}]\exp\left[\mathrm{i}\int \mathrm{d}^4x\{\mathcal{L}_{\mathrm{total}}((3.4.26)) - \tfrac{1}{2}F_\alpha^2(\{\phi_a\})\}\right]\\ &= \int \mathcal{D}[\{\phi_a\}]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[\mathrm{i}\int \mathrm{d}^4x\{\mathcal{L}_{\mathrm{total}}((3.4.26)) - \tfrac{1}{2}F_\alpha^2(\{\phi_a\})\right.\\ &\quad\left. +\bar{c}_\alpha(x)M_F(\{\phi_a(x)\})c_\beta(x)\}\right].\end{aligned} \tag{3.4.47}$$

From the fact that the gauge-fixing condition, (3.4.29), is linear with respect to $\{\phi_a\}$, we have

$$\{M_F(\{\phi_a\})\}_{\alpha x,\beta y} = \delta^4(x-y)M_F(\{\phi_a(x)\})_{\alpha,\beta}. \tag{3.4.48}$$

Summarizing the results, we have

$$\langle 0,\mathrm{out}|0,\mathrm{in}\rangle_F = \int \mathcal{D}[\{\phi_a\}]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[\mathrm{i}I_{\mathrm{eff}}[\{\phi_a\},\bar{c}_\alpha,c_\beta]\right], \tag{3.4.49a}$$

with the effective action functional $I_{\mathrm{eff}}[\{\phi_a\},\bar{c}_\alpha,c_\beta]$ given by

$$I_{\mathrm{eff}}[\{\phi_a\},\bar{c}_\alpha,c_\beta] = \int \mathrm{d}^4x\mathcal{L}_{\mathrm{eff}}(\{\phi_a(x)\},\bar{c}_\alpha(x),c_\beta(x)), \tag{3.4.49b}$$

and the effective Lagrangian density $\mathcal{L}_{\mathrm{eff}}(\{\phi_a(x)\},\bar{c}_\alpha(x),c_\beta(x))$ given by

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}(\{\phi_a(x)\},\bar{c}_\alpha(x),c_\beta(x)) &= \mathcal{L}_{\mathrm{total}}(\{\phi_a(x)\};(3.4.26)) - \tfrac{1}{2}F_\alpha^2(\{\phi_a(x)\})\\ &\quad +\bar{c}_\alpha(x)M_F(\{\phi_a(x)\})_{\alpha,\beta}c_\beta(x).\end{aligned} \tag{3.4.49c}$$

**(4) $R_\xi$-Gauge**: In order to eliminate the mixed term

$$\partial_\mu\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{v})A_\alpha'^\mu(x)$$

in the quadratic part of the total Lagrangian density $\mathcal{L}_{\mathrm{total}}^{\mathrm{quad.}}$ in (3.4.27), which couples the longitudinal component of the gauge field, $\mathrm{i}(\theta_\alpha\tilde{v})A_\alpha'^\mu(x)$, and the Higgs scalar field, $\tilde{\eta}(x)$, we choose the $R_\xi$-gauge-fixing condition as

$$F_\alpha(\{\phi_a(x)\}) = \sqrt{\xi}\left(\partial_\mu A_\alpha'^\mu(x) - \frac{1}{\xi}\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{v})\right), \qquad \xi > 0. \tag{3.4.50}$$

We have the exponent of the quasi-Gaussian functional as

$$\frac{1}{2}F_\alpha^2(\{\phi_a(x)\}) = \frac{\xi}{2}(\partial_\mu A_\alpha'^\mu(x))^2 - \tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{v})\partial_\mu A_\alpha'^\mu(x) - \frac{1}{2\xi}\{(\theta_\alpha\tilde{v})\tilde{\eta}(x)\}^2, \tag{3.4.51}$$

so that the mixed terms add up to the 4-divergence,

$$\begin{aligned}&\{\mathcal{L}_{\text{total}}^{\text{quad.}}-\tfrac{1}{2}F_\alpha^2(\{\phi_a(x)\})\}_{\text{mixed}}\\&\quad=\partial_\mu\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{v})A_\alpha'^{\mu}(x)+\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{v})\partial_\mu A_\alpha'^{\mu}(x)\\&\quad=\partial_\mu\{\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{v})A_\alpha'^{\mu}(x)\}\,,\end{aligned}\tag{3.4.52}$$

i.e., in the $R_\xi$-gauge, the mixed term does not contribute to the effective action functional $I_{\text{eff}}[\{\phi_a\},\bar{c}_\alpha,c_\beta]$.

We calculate the Faddeev–Popov determinant in the $R_\xi$-gauge from the transformation laws of $A'_{\alpha\mu}(x)$ and $\tilde{\eta}(x)$.

$$\delta A'_{\alpha\mu}(x)=-(D_\mu^{\text{adj}}\varepsilon(x))_\alpha,\qquad\alpha=1,\ldots,N.\tag{3.4.53a}$$

$$\delta\tilde{\eta}(x)=\mathrm{i}\varepsilon_\alpha(x)\{\theta_\alpha(\tilde{v}+\tilde{\eta}(x))\}.\tag{3.4.53b}$$

$$\begin{aligned}\{M_F(\{\phi_a\})\}_{\alpha x,\beta y}&\equiv\frac{\delta F_\alpha(\{\phi_a^g(x)\})}{\delta\varepsilon_\beta(y)}|_{g=1}\\&\equiv\delta^4(x-y)M_F(\{\phi_a(x)\})_{\alpha,\beta}\\&=\delta^4(x-y)\sqrt{\xi}\Big\{-\partial^\mu(D_\mu^{\text{adj}})_{\alpha,\beta}\\&\qquad-\frac{1}{\xi}\mu_{\alpha,\beta}^2-\frac{1}{\xi}(\tilde{v},\theta_\alpha\theta_\beta\tilde{\eta}(x))\Big\}\,.\end{aligned}\tag{3.4.54}$$

We absorb $\sqrt{\xi}$ into the normalization of the Faddeev–Popov ghost fields,

$$\{\bar{c}_\alpha(x),c_\beta(x)\}.$$

We then have the Faddeev–Popov ghost Lagrangian density as

$$\begin{aligned}\mathcal{L}_{\text{ghost}}(\bar{c}_\alpha(x),c_\beta(x))&\equiv\bar{c}_\alpha(x)M_F(\{\phi_a(x)\})_{\alpha,\beta}c_\beta(x)\\&=\partial_\mu\bar{c}_\alpha(x)\partial^\mu c_\alpha(x)-\frac{1}{\xi}\mu_{\alpha,\beta}^2\bar{c}_\alpha(x)c_\beta(x)\\&\quad+C_{\gamma\alpha\beta}\partial_\mu\bar{c}_\alpha(x)A_\gamma'^{\mu}(x)c_\beta(x)\\&\quad-\frac{1}{\xi}\bar{c}_\alpha(x)(\tilde{v},\theta_\alpha\theta_\beta\tilde{\eta}(x))c_\beta(x)\,.\end{aligned}\tag{3.4.55}$$

From $\mathcal{L}_{\text{total}}((3.4.26))$, the gauge-fixing term, $-\frac{1}{2}F_\alpha^2(\{\phi_a(x)\})$, (3.4.51), and $\mathcal{L}_{\text{ghost}}((3.4.55))$, we have the effective Lagrangian density $\mathcal{L}_{\text{eff}}((3.4.49c))$ as

$$\begin{aligned}\mathcal{L}_{\text{eff}}((3.4.49\text{c}))&=\mathcal{L}_{\text{total}}((3.4.26))-\frac{1}{2}F_\alpha^2(\{\phi_a(x)\})+\mathcal{L}_{\text{ghost}}((3.4.55))\\&=\mathcal{L}_{\text{eff}}^{\text{quad}}+\mathcal{L}_{\text{eff}}^{\text{int}}\,,\end{aligned}\tag{3.4.56}$$

where $\mathcal{L}_{\text{eff}}^{\text{quad}}$ and $\mathcal{L}_{\text{eff}}^{\text{int}}$ are respectively given by

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{quad}} &= -\frac{1}{2}A'_{\alpha\mu}(x)[\delta_{\alpha\beta}\{-\eta^{\mu\nu}\partial^2 + (1-\xi)\partial^\mu\partial^\nu\} - \mu^2_{\alpha,\beta}\eta^{\mu\nu}]A'_{\beta\nu}(x) \\ &\quad +\frac{1}{2}\eta_i(x)\left\{-\delta_{ij}\partial^2 - (M^2)_{ij} + \frac{1}{\xi}(\theta_\alpha\tilde{v})_i(\theta_\alpha\tilde{v})_j\right\}\eta_j(x) \\ &\quad +\bar{c}_\alpha(x)\left\{-\delta_{\alpha\beta}\partial^2 - \frac{1}{\xi}\mu^2_{\alpha,\beta}\right\}c_\beta(x)\,, \end{aligned} \tag{3.4.57}$$

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{int}} &= \mathcal{L}_{\text{eff}} - \mathcal{L}_{\text{eff}}^{\text{quad}} = \frac{1}{2}C_{\alpha\beta\gamma}A'_{\beta\mu}(x)A'_{\gamma\nu}(x)(\partial^\mu A'^\nu_\alpha(x) - \partial^\nu A'^\mu_\alpha(x)) \\ &\quad -\frac{1}{4}C_{\alpha\beta\gamma}C_{\alpha\delta\varepsilon}A'_{\beta\mu}(x)A'_{\gamma\nu}(x)A'^\mu_\delta(x)A'^\nu_\varepsilon(x) + \partial_\mu\tilde{\eta}(x)\mathrm{i}(\theta_\alpha\tilde{\eta}(x))A'^\mu_\alpha(x) \\ &\quad +\tilde{\eta}(x)(\theta_\alpha\theta_\beta\tilde{v})A'_{\alpha\mu}(x)A'^\mu_\alpha(x) - \frac{1}{3!}f_{ijk}\eta_i(x)\eta_j(x)\eta_k(x) \\ &\quad +\frac{1}{2}\tilde{\eta}(x)(\theta_\alpha\theta_\beta\tilde{\eta}(x))A'_{\alpha\mu}(x)A'^\mu_\beta(x) - \frac{1}{4!}f_{ijkl}\eta_i(x)\eta_j(x)\eta_k(x)\eta_l(x) \\ &\quad +C_{\alpha\beta\gamma}\partial_\mu\bar{c}_\alpha(x)c_\beta(x)A'^\mu_\gamma(x) - \frac{1}{\xi}\bar{c}_\alpha(x)(\tilde{v},\theta_\alpha\theta_\beta\tilde{\eta}(x))c_\beta(x). \end{aligned} \tag{3.4.58}$$

From $\mathcal{L}_{\text{eff}}^{\text{quad}}$((3.4.57)), we have the equations satisfied by the "free" Green's functions,

$$\{D^{(A')}_{\alpha\mu,\beta\nu}(x-y), D^{(\eta)}_{i,j}(x-y), D^{(C)}_{\alpha,\beta}(x-y)\},$$

of the gauge fields, the Higgs scalar fields and the Faddeev–Popov ghost fields,

$$\{A'_{\alpha\mu}(x); \eta_i(x); \bar{c}_\alpha(x); c_\beta(x)\},$$

as

$$\begin{aligned}&-[\delta_{\alpha\beta}\{(-\eta^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu) - \xi\partial^\mu\partial^\nu\} - \mu^2_{\alpha,\beta}\eta^{\mu\nu}]D^{(A')}_{\beta\nu,\gamma\lambda}(x-y) \\ &\quad = \delta_{\alpha\gamma}\eta^\mu_\lambda\delta^4(x-y),\end{aligned} \tag{3.4.59a}$$

$$\left\{-\delta_{ij}\partial^2 - (M^2)_{ij} + \frac{1}{\xi}(\theta_\alpha\tilde{v})_i(\theta_\alpha\tilde{v})_j\right\}D^{(\eta)}_{j,k}(x-y) = \delta_{ik}\delta^4(x-y), \tag{3.4.59b}$$

$$\left\{-\delta_{\alpha\beta}\partial^2 - \frac{1}{\xi}\mu^2_{\alpha,\beta}\right\}D^{(C)}_{\beta,\gamma}(x-y) = \delta_{\alpha\gamma}\delta^4(x-y). \tag{3.4.59c}$$

Fourier transforming the "free" Green's functions as

$$\begin{pmatrix} D^{(A')}_{\alpha\mu,\beta\nu}(x-y) \\ D^{(\eta)}_{i,j}(x-y) \\ D^{(C)}_{\alpha,\beta}(x-y) \end{pmatrix} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\exp[-\mathrm{i}k(x-y)] \begin{pmatrix} D^{(A')}_{\alpha\mu,\beta\nu}(k) \\ D^{(\eta)}_{i,j}(k) \\ D^{(C)}_{\alpha,\beta}(k) \end{pmatrix}, \quad (3.4.60)$$

we have the equations satisfied by the momentum space "free" Green's functions as

$$-\left[\delta_{\alpha\beta}k^2\left\{\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)+\xi\frac{k^\mu k^\nu}{k^2}\right\}-\mu^2_{\alpha,\beta}\eta^{\mu\nu}\right]D^{(A')}_{\beta\nu,\gamma\lambda}(k)=\delta_{\alpha\gamma}\eta^{\mu}_{\lambda}, \quad (3.4.61a)$$

$$\left\{\delta_{ij}k^2-(M^2)_{ij}+\frac{1}{\xi}(\theta_\alpha\tilde{v})_i(\theta_\alpha\tilde{v})_j\right\}D^{(\eta)}_{j,k}(k)=\delta_{ik}, \quad (3.4.61b)$$

$$\left\{\delta_{\alpha\beta}k^2-\frac{1}{\xi}\mu^2_{\alpha,\beta}\right\}D^{(C)}_{\beta,\gamma}(k)=\delta_{\alpha\gamma}. \quad (3.4.61c)$$

We obtain the momentum space "free" Green's functions as

$$\begin{aligned} D^{(A')}_{\alpha\mu,\beta\nu}(k) &= -\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\left(\frac{1}{k^2-\mu^2}\right)_{\alpha,\beta}-\frac{1}{\xi}\frac{k_\mu k_\nu}{k^2}\left(\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta} \\ &= -\eta_{\mu\nu}\left(\frac{1}{k^2-\mu^2}\right)_{\alpha,\beta}-\left(\frac{1}{\xi}-1\right)k_\mu k_\nu\left(\frac{1}{k^2-\mu^2}\cdot\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta}, \end{aligned} \quad (3.4.62a)$$

$$\begin{aligned} D^{(\eta)}_{i,j}(k) &= (1-P)_{i,k}\left(\frac{1}{k^2-M^2}\right)_{k,j} \\ &\quad -(\theta_\alpha\tilde{v})_i\left(\frac{1}{\mu^2}\cdot\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta}(\theta_\beta\tilde{v})_j \\ &= \left(\frac{1}{k^2-M^2}\right)_{i,j}-(\theta_\alpha\tilde{v})_i\frac{1}{\xi}\cdot\frac{1}{k^2}\cdot\left(\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta}(\theta_\beta\tilde{v})_j, \end{aligned} \quad (3.4.62b)$$

$$D^{(C)}_{\alpha,\beta}(k)=\left(\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta}. \quad (3.4.62c)$$

Here, $P_{i,j}$ is the projection operator onto the $(N-M)$-dimensional subspace spanned by the Nambu–Goldstone boson:

$$P_{i,j}=\sum_{\theta_\alpha,\theta_\beta\notin S}(\theta_\alpha\tilde{v})_i\left(\frac{1}{\mu^2}\right)_{\alpha,\beta}(\theta_\beta\tilde{v})^\dagger_j, \qquad i,j=1,\ldots,n, \quad (3.4.63a)$$

$$P_{i,j}(\theta_\gamma \tilde{v})_j = (\theta_\gamma \tilde{v})_i, \quad (\theta_\gamma \tilde{v})_i^\dagger P_{i,j} = (\theta_\gamma \tilde{v})_j^\dagger, \quad \theta_\gamma \notin S. \tag{3.4.63b}$$

The $R_\xi$-gauge is not only the gauge which eliminates the mixed term of $A'_{\alpha\mu}(x)$ and $\tilde{\eta}(x)$ in the quadratic part of the effective Lagrangian density, but also the gauge which connects the unitarity gauge ($\xi = 0$), the $'$t Hooft–Feynman gauge ($\xi = 1$) and the Landau gauge ($\xi \to \infty$) continuously in $\xi$.

**(4.1) Unitarity Gauge** ($\xi = 0$):

$$D^{(A')}_{\alpha\mu,\beta\nu}(k) = -\left\{\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{\mu^2}\right) \cdot \left(\frac{1}{k^2 - \mu^2}\right)\right\}_{\alpha,\beta}, \tag{3.4.64a}$$

$$D^{(\eta)}_{i,j}(k) = \left(\frac{1}{k^2 - M^2}\right)_{i,j} + \frac{1}{k^2}(\theta_\alpha \tilde{v})_i \cdot \left(\frac{1}{\mu^2}\right)_{\alpha\beta} \cdot (\theta_\beta \tilde{v})_j, \tag{3.4.64b}$$

$$D^{(C)}_{\alpha,\beta}(k) \propto \xi \to 0, \qquad \text{infinitely massive ghost.} \tag{3.4.64c}$$

**(4.2) $'$t Hooft–Feynman Gauge** ($\xi = 1$):

$$D^{(A')}_{\alpha\mu,\beta\nu}(k) = -\eta_{\mu\nu}\left(\frac{1}{k^2 - \mu^2}\right)_{\alpha,\beta}, \tag{3.4.65a}$$

$$D^{(\eta)}_{i,j}(k) = \left(\frac{1}{k^2 - M^2}\right)_{i,j} - (\theta_\alpha \tilde{v})_i \frac{1}{k^2}\left(\frac{1}{k^2 - \mu^2}\right)_{\alpha\beta}(\theta_\beta \tilde{v})_j, \tag{3.4.65b}$$

$$D^{(C)}_{\alpha,\beta}(k) = \left(\frac{1}{k^2 - \mu^2}\right)_{\alpha,\beta}, \qquad \text{massive ghost at } k^2 = \mu^2. \tag{3.4.65c}$$

**(4.3) Landau Gauge** ($\xi \to \infty$):

$$D^{(A')}_{\alpha\mu,\beta\nu}(k) = -\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\left(\frac{1}{k^2 - \mu^2}\right)_{\alpha,\beta}, \tag{3.4.66a}$$

$$D^{(\eta)}_{i,j}(k) = \left(\frac{1}{k^2 - M^2}\right)_{i,j}, \tag{3.4.66b}$$

$$D^{(C)}_{\alpha,\beta}(k) = \frac{\delta_{\alpha\beta}}{k^2}, \qquad \text{massless ghost at } k^2 = 0. \tag{3.4.66c}$$

We have the generating functional of (the connected part of) the "full" Green's functions as

$$
\begin{aligned}
Z_F[\{J_a\},\zeta_\alpha,\bar{\zeta}_\beta] &\equiv \exp[\mathrm{i}W_F[\{J_a\},\zeta_\alpha,\bar{\zeta}_\beta]] \\
&\equiv \left\langle 0,\mathrm{out}\left|\mathrm{T}\left(\exp\left[\mathrm{i}\int \mathrm{d}^4z\{J_a(z)\hat{\phi}_a(z)\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. + \bar{c}_\alpha(z)\zeta_\alpha(z) + \bar{\zeta}_\beta(z)c_\beta(z)\}\right]\right)\right|0,\mathrm{in}\right\rangle \\
&= \int \mathcal{D}[\phi_a]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[\mathrm{i}\int \mathrm{d}^4z\{\mathcal{L}_{\mathrm{eff}}(\{\phi_a(z)\},\bar{c}_\alpha(z),c_\beta(z))\right. \\
&\qquad \left. +J_a(z)\phi_a(z) + \bar{c}_\alpha(z)\zeta_\alpha(z) + \bar{\zeta}_\beta c_\beta(z)\}\right] \\
&= \exp\left[\mathrm{i}I_{\mathrm{eff}}^{\mathrm{int}}\left[\left\{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_a}\right\},\mathrm{i}\frac{\delta}{\delta\zeta_\alpha},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\zeta}_\beta}\right]\right] Z_{F,0}\left[\{J_a\},\zeta_\alpha,\bar{\zeta}_\beta\right], \qquad (3.4.67)
\end{aligned}
$$

with

$$
\begin{aligned}
Z_{F,0}[\{J_a\},\zeta_\alpha,\bar{\zeta}_\beta] &= \int \mathcal{D}[\phi_a]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[\mathrm{i}\int \mathrm{d}^4z\{\mathcal{L}_{\mathrm{eff}}^{\mathrm{quad}}((3.4.57))\right. \\
&\qquad \left. +J_a(z)\phi_a(z) + \bar{c}_\alpha(z)\zeta_\alpha(z) + \bar{\zeta}_\beta(z)c_\beta(z)\}\right] \\
&= \exp\left[\mathrm{i}\int \mathrm{d}^4x\mathrm{d}^4y\{-\tfrac{1}{2}J_\alpha^\mu(x)D_{\alpha\mu,\beta\nu}^{(A')}(x-y)J_\beta^\nu(y)\right. \\
&\qquad \left. -\tfrac{1}{2}J_i(x)D_{i,j}^{(\eta)}(x-y)J_j(y) - \bar{\zeta}_\beta(x)D_{\beta,\alpha}^{(C)}(x-y)\zeta_\alpha(y)\}\right], \qquad (3.4.68)
\end{aligned}
$$

and

$$
I_{\mathrm{eff}}^{\mathrm{int}}[\{\phi_a\},\bar{c}_\alpha,c_\beta] = \int \mathrm{d}^4x\mathcal{L}_{\mathrm{eff}}^{\mathrm{int}}((3.4.58)). \qquad (3.4.69)
$$

Since the Faddeev–Popov ghost, $\{\bar{c}_\alpha(x), c_\beta(x)\}$, appears only in the internal loop, we might as well set

$$
\zeta_\alpha(z) = \bar{\zeta}_\beta(z) = 0
$$

in $Z_F[\{J_a\},\zeta_\alpha,\bar{\zeta}_\beta]$, (3.4.67), and define $Z_F[\{J_a\}]$ by

$$
\begin{aligned}
Z_F[\{J_a\}] &\equiv \exp[\mathrm{i}W_F[\{J_a\}]] \\
&\equiv \left\langle 0,\mathrm{out}\left|\mathrm{T}\left(\exp\left[\mathrm{i}\int \mathrm{d}^4zJ_a(z)\hat{\phi}_a(z)\right]\right)\right|0,\mathrm{in}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \int \mathcal{D}[\phi_a]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta] \\
&\quad \times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\mathrm{eff}}(\{\phi_a(z)\},\bar{c}_\alpha(z),c_\beta(z)) + J_a(z)\phi_a(z)\}\right] \\
&= \int \mathcal{D}[\phi_a]\Delta_F[\{\phi_a\}] \\
&\quad \times \exp\left[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\mathrm{total}}(\{\phi_a(z)\}) - \tfrac{1}{2}F_\alpha^2(\{\phi_a\}) + J_a(z)\phi_a(z)\}\right] \\
&= Z_F\left[\{J_a\},\zeta_\alpha = 0,\bar{\zeta}_\beta = 0\right], \qquad (3.4.70)
\end{aligned}
$$

which we shall use in the next section.

### 3.4.4 Ward–Takahashi–Slavnov–Taylor Identity and the $\xi$-Independence of the Physical $S$-Matrix

In the functional integrand of $Z[\{J_a\}]$, (3.4.70), we perform the nonlinear gauge transformation $g_0$, (3.4.40). Then, by an argument identical to Sect. 3.3.4, we obtain the Ward–Takahashi–Slavnov–Taylor identity

$$
\begin{aligned}
&\Bigg\{ - F_\alpha\left(\left\{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_a}\right\}\right) + J_a\left(\mathrm{i}\Gamma_{ab}^{\beta}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_b} + \Lambda_a^{\beta}\right) \\
&\qquad \times \left(M_F^{-1}\left(\left\{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_a}\right\}\right)\right)_{\beta,\alpha}\Bigg\} Z_F[\{J_a\}] = 0, \qquad (3.4.71)
\end{aligned}
$$

where $\Gamma_{ab}^{\alpha}$ and $\Lambda_a^{\alpha}$ are defined by (A4.3) in Appendix 4. Armed with this identity, we move on to prove the $\xi$-independence of the physical $S$-matrix. By an argument entirely similar to that of Sect. 3.3.4, under an infinitesimal variation of the gauge-fixing condition,

$$
F_\alpha(\{\phi_a(x)\}) \to F_\alpha(\{\phi_a(x)\}) + \Delta F_\alpha(\{\phi_a(x)\}), \qquad (3.4.72)
$$

with the use of the Ward–Takahashi–Slavnov–Taylor identity, (3.4.71), we obtain $Z_{F+\Delta F}[\{J_a\}]$ to first order in $\Delta F(\{\phi_a(x)\})$ as

$$
\begin{aligned}
Z_{F+\Delta F}[\{J_a\}] &= \int \mathcal{D}[\phi_a]\Delta_F[\{\phi_a\}] \\
&\quad \times \exp\Bigg[\mathrm{i}\int \mathrm{d}^4 z\{\mathcal{L}_{\mathrm{total}}(\{\phi_a(z)\}) \\
&\quad -\frac{1}{2}F_\alpha^2(\{\phi_a(z)\}) + J_a(z)\phi_a^{\tilde{g}_0^{-1}}(z)\}\Bigg], \qquad (3.4.73)
\end{aligned}
$$

with

$$\phi_a^{\tilde{g}_0^{-1}}(z) = \phi_a(z) - (\mathrm{i}\Gamma_{ab}^{\alpha}\phi_b + \Lambda_a^{\alpha})(M_F^{-1}(\{\phi_b(z)\}))_{\alpha,\beta}\Delta F_\beta(\{\phi_b(z)\}). \tag{3.4.74}$$

The transformation, $\tilde{g}_0$, is the gauge transformation generated by the parameter

$$\varepsilon_\alpha(x; M_F(\{\phi_a(x)\}), \Delta F_\beta(\{\phi_a(x)\})) = (M_F^{-1}(\{\phi_a(x)\}))_{\alpha,\beta}\Delta F_\beta(\{\phi_a(x)\}). \tag{3.4.75}$$

The response of the gauge-fixing condition, $F_\alpha(\{\phi_a(x)\})$, to this gauge transformation $\tilde{g}_0$ provides (3.4.72). In this manner, the variation of the gauge-fixing condition, (3.4.72), gives rise to an extra vertex,

$$J_a(x)(\phi_a^{\tilde{g}_0^{-1}}(x) - \phi_a(x)) \tag{3.4.76}$$

between the external hook $\{J_a(x)\}$ and the matter-gauge system. We can take care of this extra vertex by the renormalization of the wave function renormalization constants of the external lines of the $S$-matrix on the mass-shell. Namely, we have

$$S_{F+\Delta F} = \prod_e (Z_{F+\Delta F}/Z_F)_e^{1/2} S_F. \tag{3.4.77}$$

From (3.4.77), we find that the renormalized $S$-matrix, $S_{\text{ren.}}$, defined by

$$S_{\text{ren.}} = S_F/\prod_e \left(Z_F^{1/2}\right)_e \tag{3.4.78}$$

does not depend on $F_\alpha(\{\phi_a(x)\})$, i.e., $S_{\text{ren.}}$ is gauge independent. We have thus verified that $S_{\text{ren.}}$ is $\xi$-independent in the $R_\xi$-gauge. In this respect, the $R_\xi$-gauge is a convenient gauge-fixing condition. Without fixing the $\xi$-parameter at $\xi = 0$, 1, or $\infty$, we can use the cancelation of the $\xi$-dependent terms, originating from the spurious pole at $k^2 = \mu^2/\xi$ and the $\xi$-dependent vertex of $\mathcal{L}_{\text{eff}}^{\text{int.}}$((3.4.58)) in the perturbative calculation, as a check of the accuracy of the calculation.

## 3.5 Bibliography

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**Chapter 3. Section 3.1**

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$$\begin{cases} \textit{Strong} & \text{by } \textit{Weak}, \\ \textit{Isotopic Spin} & \text{by } \textit{Weak Isotopic Spin}, \\ \textit{Hypercharge} & \text{by } \textit{Weak Hypercharge}, \end{cases}$$

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**Gravitational Field from the Gauge Principle.** It is anachronistic to say that the gravitational field is also a gauge field. Indeed, the gravitational field is *the genesis of the gauge field* as originally conceived by H. Weyl before the advent of the correct Weyl's gauge principle. The Lagrangian density for the gravitational field can be derived from the requirement of the gauge invariance of the action functional $I[\phi_a]$ under the local 10-parameter Poincare transformation. Ordinarily, it is obtained from the principle of equivalence.

We cite the following books as a general reference on the theory of the gravitational field.

(**T-52**) Weinberg, S.; "Gravitation and Cosmology. Principles and Applications of The General Theory of Relativity", John Wiley and Sons, Inc., 1972, New York.
(**T-53**) Low, F.E.; "Classical Field Theory", John Wiley and Sons, Inc., 1997, New York.

We cite the following articles for the discussion of the gravitational field in terms of the gauge principle.

(**R3-5**) Utiyama, R.; Phys. Rev. **101**, (1956), 1597, Section 4.
(**R3-54**) Kibble, T.W.B.; J. Math. Phys. **2**, (1961), 212.

We cite the following articles for path integral quantization of the gravitational field.

(**R3-15**) Popov, V.N. and Faddeev, L.D.; Kiev report. No.ITP-67-36, (1967); translated into English by B.W. Lee and D. Gordon, NAL-THY-57, (1972), Section V.

(**R3-55**) Faddeev, L.D. and Popov, V.N.; Sov. Phys.-Usp. **16**, (1974), 777.

We cite the following book and the references cited therein for *Supersymmetry* and *Supergravity.*

(**T-16**) Weinberg, S.; ″Quantum Theory of Fields III″, Cambridge Univ. Press, 2000, New York. The entire book.

We cite the following book as an introduction to *Superstrings* in the path integral formalism.

(**T-54**) Kaku, M.; ″Introduction to Superstrings″, Springer-Verlag, 1988, New York.

# 4. Path Integral Representation of Quantum Statistical Mechanics

In this chapter, we discuss the path integral representation of quantum statistical mechanics with the following two methods.

(1) The *Analytic continuation* of the results of Chaps. 1 and 2 to negative imaginary time,

$$t \to -\mathrm{i}\tau, \qquad 0 \le \tau \le \beta \equiv \frac{1}{k_{\mathrm{B}}T} .$$

(2) The *Fradkin construction* starting from the canonical formalism based on the extended Matsubara representation of quantum field theory at finite temperature.

In Sect. 4.1, we discuss the method of analytic continuation heuristically. As a result of the analytic continuation of the path integral representation of quantum mechanics discussed in Chap. 1, we obtain the path integral representation of the partition function of the canonical ensemble at finite temperature in which the total number $N$ of particles is fixed (Sect. 4.1.1). As a result of the analytic continuation of the path integral representation of quantum field theory discussed in Chap. 2, we obtain the partition function of the grand canonical ensemble at finite temperature in which the total number $N$ of particles varies (Sect. 4.1.2). In both approaches, we employ the Lagrangian formalism.

In Sect. 4.2, based on the extended Matsubara representation of quantum field theory at finite temperature, with the standard method of the introduction of the interaction picture in the canonical formalism, we derive a functional differential equation of the partition function of the grand canonical ensemble, $Z_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta])$, in the presence of external hooks $\{J(x), \bar{\eta}(x), \eta(x)\}$ (Sect. 4.2.1). Following the procedure used in the Symanzik construction in Sect. 2.3, we obtain the path integral representation of the partition function of the grand canonical ensemble, $Z_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta])$ (Sect. 4.2.2).

In Sect. 4.3, we consider a relativistic interacting fermion–boson system. As a by-product of the Fradkin construction, we derive the Schwinger–Dyson equation for the "full" temperature Green's functions of the system from the functional differential equation for $Z_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta])$ (Sect. 4.3.1). We confirm the agreement of the nonrelativistic limit of the fermion "free" temperature

M. Masujima: Path Integral Quantization and Stochastic Quantization,
STMP 165, 175–207 (2000)

Green's function with a nonrelativistic treatment of the fermion "free" temperature Green's function derived in Sect. 4.1 (Sect. 4.3.2).

In Sect. 4.4, we discuss the linearization of the four-fermion interaction with the method of the auxiliary field which frequently shows up in the model theory of condensed matter physics. By linearization, we mean the tri-linear Yukawa coupling of the fermion fields, $\psi(x)$ and $\bar{\psi}(x)$, with the auxiliary bosonic field $\phi(x)$. Mathematically, we simply incomplete the square in the exponent by performing the inverse of Gaussian integration. We get a new perspective by changing the form of the interaction. We can reduce the four-fermion interaction to bi-linear form in the fermion fields and thus to the calculation of the fermion determinant which depends on the auxiliary field $\phi(x)$ (Sect. 4.4.1). In the method of the auxiliary field in the Lagrangian formalism, we treat the auxiliary field $\phi(x)$ on an equal footing with the physical fields, $\psi(x)$ and $\bar{\psi}(x)$, and find that the "full" Green's function of $\phi(x)$ is related to the linear response function of the system. As the method of the auxiliary field in the Hamiltonian formalism, we have the Stratonovich–Hubbard transformation (Sect. 4.4.2). In this method, we use the Gaussian auxiliary field $z(\tau)$. Mathematically, $z(\tau)$ resembles the auxiliary field $\phi(x)$ in the Lagrangian formalism. But the physical content of the Gaussian auxiliary field $z(\tau)$ is not so clear as compared to the auxiliary field $\phi(x)$ in the Lagrangian formalism.

As for Feynman's variational principle in statistical mechanics and its application to the polaron problem, the reader is referred to Feynman and Hibbs.

## 4.1 Partition Function of the Canonical Ensemble and the Grand Canonical Ensemble

In this section, we obtain the path integral representation of the partition functions, $Z_{\mathrm{C}}(\beta)$ and $Z_{\mathrm{GC}}(\beta)$, of the canonical ensemble and the grand canonical ensemble by analytic continuation of the various formulas obtained for quantum mechanics in Chap. 1 and quantum field theory in Chap. 2. We defer the rigorous derivation of these results, due to E.S. Fradkin, to Sect. 4.2, which is based on the extended Matsubara representation of quantum field theory at finite temperature.

In Sect. 4.1.1, we derive the path integral representation of the partition function, $Z_{\mathrm{C}}(\beta)$, of the canonical ensemble in terms of the *Euclidean Lagrangian*,

$$L_{\mathrm{E}}\left(\left\{\boldsymbol{q}_j(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{q}_j(\tau)\right\}_{j=1}^{N}\right),$$

based on the observation that the Bloch equation satisfied by the density matrix $\hat{\rho}_{\mathrm{C}}(\tau)$ of the canonical ensemble is equivalent to the imaginary time Schrödinger equation under analytic continuation,

$$t=-\mathrm{i}\tau, \qquad 0\le\tau\le\beta\equiv\frac{1}{k_{\mathrm{B}}T}\ .$$

In Sect. 4.1.2, we extend the consideration to the grand canonical ensemble in terms of the *modified Euclidean Lagrangian density*,

$$\begin{aligned}&\mathcal{L}'_{\mathrm{E}}(\psi_\alpha(\tau,\boldsymbol{x}),\partial_\mu\psi_\alpha(\tau,\boldsymbol{x}))\\ &\quad=\mathcal{L}_{\mathrm{E}}(\psi_\alpha(\tau,\boldsymbol{x}),\partial_\mu\psi_\alpha(\tau,\boldsymbol{x}))+\mu\psi^\dagger_\alpha(\tau,\boldsymbol{x})\psi_\alpha(\tau,\boldsymbol{x})\ ,\end{aligned}$$

and obtain the path integral representation of the partition function, $Z_{\mathrm{GC}}(\beta)$, of the grand canonical ensemble. Based on these considerations, we introduce the "full" temperature Green's function and also obtain the path integral representation of the generating functional, $Z_{\mathrm{GC}}[\beta; J_\alpha, J^\dagger_\beta]$, of the "full" temperature Green's functions.

### 4.1.1 The Canonical Ensemble and the Bloch Equation

We consider the canonical ensemble described by the Hamiltonian

$$\hat{H}(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)$$

at finite temperature,

$$\beta=1/k_{\mathrm{B}}T, \quad k_{\mathrm{B}}=\text{Boltzman constant}, \quad T=\text{absolute temperature}\ .$$

The density matrix $\hat{\rho}_{\mathrm{C}}(\beta)$ of this system satisfies the Bloch equation,

$$-\frac{\partial}{\partial\tau}\hat{\rho}_{\mathrm{C}}(\tau)=\hat{H}(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)\hat{\rho}_{\mathrm{C}}(\tau)\ , \qquad 0\le\tau\le\beta\ , \tag{4.1.1}$$

with its formal solution given by

$$\hat{\rho}_{\mathrm{C}}(\tau)=\exp[-\tau\hat{H}(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N]\hat{\rho}_{\mathrm{C}}(0). \tag{4.1.2}$$

We compare the Bloch equation and the density matrix, (4.1.1) and (4.1.2), with the Schrödinger equation for the state vector $|\psi,t\rangle$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi,t\rangle=\hat{H}(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)|\psi,t\rangle\ , \tag{4.1.3}$$

and its formal solution given by

$$|\psi,t\rangle=\exp[-\mathrm{i}t\hat{H}(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)]|\psi,0\rangle\ . \tag{4.1.4}$$

We find that by analytic continuation,

$$t=-\mathrm{i}\tau, \qquad 0\le\tau\le\beta\ , \tag{4.1.5}$$

the (real time) Schrödinger equation and its formal solution, (4.1.3) and (4.1.4), are analytically continued into the Bloch equation and the density matrix, (4.1.1) and (4.1.2), respectively. Under analytic continuation, (4.1.5), we divide the interval $[0,\beta]$ into $n$ equal subintervals, and use the resolution of the identity in both the $q$-representation and the $p$-representation. In this way, we obtain the following list of correspondences. Here, we assume the Hamiltonian $\hat{H}(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)$ of the following form,

$$\hat{H}(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)=\sum_{j=1}^{N}\frac{1}{2m}\boldsymbol{p}_j^2+\sum_{j\rangle k}V(\boldsymbol{q}_j,\boldsymbol{q}_k)\ . \tag{4.1.6}$$

**List of Correspondences**

| **Quantum mechanics** | **Quantum statistical mechanics** |
|---|---|
| **Schrödinger equation**<br>$\mathrm{i}\frac{\partial}{\partial t}\lvert\psi,t\rangle$<br>$=H(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)\lvert\psi,t\rangle.$ | **Bloch equation**<br>$-\frac{\partial}{\partial\tau}\hat{\rho}_{\mathrm{C}}(\tau)$<br>$=H(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)\hat{\rho}_{\mathrm{C}}(\tau).$ |
| **Schrödinger state vector**<br>$\lvert\psi,t\rangle$<br>$=\exp[-\mathrm{i}tH(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)]\lvert\psi,0\rangle.$ | **Density matrix**<br>$\hat{\rho}_{\mathrm{C}}(\tau)$<br>$=\exp[-\tau H(\{\boldsymbol{q}_j,\boldsymbol{p}_j\}_{j=1}^N)]\hat{\rho}_{\mathrm{C}}(0).$ |
| **Minkowskian Lagrangian**<br>$L_{\mathrm{M}}(\{q_j(t),\dot{q}_j(t)\}_{j=1}^N)$<br>$=\sum_{j=1}^N\frac{m}{2}\dot{q}_j^2(t)$<br>$-\sum_{j>k}^N V(\boldsymbol{q}_j,\boldsymbol{q}_k).$ | **Euclidean Lagrangian**<br>$L_{\mathrm{E}}(\{q_j(\tau),\dot{q}_j(\tau)\}_{j=1}^N)$<br>$=-\sum_{j=1}^N\frac{m}{2}\dot{q}_j^2(\tau)$<br>$-\sum_{j>k}^N V(\boldsymbol{q}_j,\boldsymbol{q}_k).$ |
| i × **Minkowskian action functional**<br>$\mathrm{i}I_{\mathrm{M}}[\{\boldsymbol{q}_j\}_{j=1}^N;\boldsymbol{q}_f,\boldsymbol{q}_i]$<br>$=\mathrm{i}\int_{t_i}^{t_f}\mathrm{d}tL_{\mathrm{M}}(\{q_j(t),\dot{q}_j(t)\}_{j=1}^N).$ | **Euclidean action functional**<br>$I_{\mathrm{E}}[\{\boldsymbol{q}_j\}_{j=1}^N;\boldsymbol{q}_f,\boldsymbol{q}_i]$<br>$=\int_0^\beta\mathrm{d}\tau L_{\mathrm{E}}(\{q_j(\tau),\dot{q}_j(\tau)\}_{j=1}^N).$ |
| **Transformation function**<br>$\langle\boldsymbol{q}_f,t_f\lvert\boldsymbol{q}_i,t_i\rangle$<br>$=\int_{\boldsymbol{q}(t_i)=\boldsymbol{q}_i}^{\boldsymbol{q}(t_f)=\boldsymbol{q}_f}\mathcal{D}[\boldsymbol{q}]$<br>$\times\exp[\mathrm{i}I_{\mathrm{M}}[\{\boldsymbol{q}_j\}_{j=1}^N;\boldsymbol{q}_f,\boldsymbol{q}_i]].$ | **Transformation function**<br>$Z_{f,i}$<br>$=\int_{\boldsymbol{q}(0)=\boldsymbol{q}_i}^{\boldsymbol{q}(\beta)=\boldsymbol{q}_f}\mathcal{D}[\boldsymbol{q}]\times$<br>$\times\exp[I_{\mathrm{E}}[\{\boldsymbol{q}_j\}_{j=1}^N;\boldsymbol{q}_f,\boldsymbol{q}_i]].$ |
| **Vacuum-to-vacuum transition amplitude**<br>$\langle 0,\mathrm{out}\lvert 0,\mathrm{in}\rangle$<br>$=\int\mathcal{D}[\boldsymbol{q}]\exp[\mathrm{i}I_{\mathrm{M}}[\{\boldsymbol{q}_j\}_{j=1}^N]].$ | **Partition function***<br>$Z_{\mathrm{C}}(\beta)=\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)$<br>$=$ "$\int\mathrm{d}\boldsymbol{q}_f\mathrm{d}\boldsymbol{q}_i\delta(\boldsymbol{q}_f-\boldsymbol{q}_i)Z_{f,i}$". |
| **Vacuum expectation value**<br>$\langle O(\hat{q})\rangle$<br>$=\frac{\int\mathcal{D}[\boldsymbol{q}]O(\boldsymbol{q})\exp[\mathrm{i}I_{\mathrm{M}}[\{\boldsymbol{q}_j\}_{j=1}^N]]}{\int\mathcal{D}[\boldsymbol{q}]\exp\{\mathrm{i}I_{\mathrm{M}}[\{\boldsymbol{q}_j\}_{j=1}^N]\}}.$ | **Thermal expectation value***<br>$\langle O(\hat{q})\rangle$<br>$=\frac{\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)O(\hat{q})}{\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)}.$ |

In this list, the entries with "*" have precise expressions given by:

**Partition function:**

$$\begin{aligned}
&Z_{\mathrm{C}}(\beta)\\
&= \mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta) = \text{“}\int \mathrm{d}^3\boldsymbol{q}_f \mathrm{d}^3\boldsymbol{q}_i \delta^3(\boldsymbol{q}_f - \boldsymbol{q}_i) Z_{f,i}\text{”}\\
&= \frac{1}{N!}\sum_P \delta_P \int \mathrm{d}^3\boldsymbol{q}_f \mathrm{d}^3\boldsymbol{q}_i \delta^3(\boldsymbol{q}_f - \boldsymbol{q}_{Pi}) Z_{f,Pi}\\
&= \frac{1}{N!}\sum_P \delta_P \int \mathrm{d}^3\boldsymbol{q}_f \mathrm{d}^3\boldsymbol{q}_{Pi} \delta^3(\boldsymbol{q}_f - \boldsymbol{q}_{Pi}) \int_{\boldsymbol{q}(0)=\boldsymbol{q}_{Pi}}^{\boldsymbol{q}(\beta)=\boldsymbol{q}_f} \mathcal{D}[\boldsymbol{q}]\\
&\quad \times \exp\left[I_{\mathrm{E}}[\{\boldsymbol{q}_j\}_{j=1}^N; \boldsymbol{q}_f, \boldsymbol{q}_{Pi}]\right] ,
\end{aligned} \tag{4.1.7}$$

and

**Thermal expectation value:**

$$\begin{aligned}
&\langle \hat{O}(\boldsymbol{q})\rangle\\
&= \frac{\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)\hat{O}(\hat{q})}{\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)}\\
&= \frac{\text{“}\int \mathrm{d}^3\boldsymbol{q}_f \mathrm{d}^3\boldsymbol{q}_i \delta^3(\boldsymbol{q}_f - \boldsymbol{q}_i) Z_{f,i}\langle i|O(\boldsymbol{q})|f\rangle\text{”}}{\int \mathrm{d}^3\boldsymbol{q}_f \mathrm{d}^3\boldsymbol{q}_i \delta^3(\boldsymbol{q}_f - \boldsymbol{q}_i) Z_{f,i}}\\
&= \frac{1}{Z_{\mathrm{C}}(\beta)}\frac{1}{N!}\sum_P \delta_P \int \mathrm{d}^3\boldsymbol{q}_f \mathrm{d}^3\boldsymbol{q}_{Pi} \delta^3(\boldsymbol{q}_f - \boldsymbol{q}_{Pi}) Z_{f,Pi}\langle \boldsymbol{q}_{Pi}|\hat{O}(\boldsymbol{q})|\boldsymbol{q}_f\rangle .
\end{aligned} \tag{4.1.8}$$

Here, $\boldsymbol{q}_i$ and $\boldsymbol{q}_f$ represent the initial position $\{\boldsymbol{q}_j(0)\}_{j=1}^N$ and the final position $\{\boldsymbol{q}_j(\beta)\}_{j=1}^N$ of $N$ identical particles, $P$ represents the permutation of $\{1, \ldots, N\}$, $Pi$ represents the permutation of the initial position $\{\boldsymbol{q}(0)\}_{j=1}^N$ and $\delta_P$ represents the signature of the permutation $P$, respectively.

$$\delta_P = \begin{cases} 1, & P \text{ even and odd}, \quad N \text{ identical bosons},\\ 1, & P \text{ even}, \quad N \text{ identical fermions},\\ -1, & P \text{ odd}, \quad N \text{ identical fermions}. \end{cases} \tag{4.1.9}$$

In this manner, we obtain the path integral representation of the partition function, $Z_{\mathrm{C}}(\beta)$, and the thermal expectation value, $\langle \hat{O}(\boldsymbol{q})\rangle$, from the formula derived in Sect. 1.2, by analytic continuation to negative imaginary time, (4.1.5).

### 4.1.2 Extension to the Grand Canonical Ensemble

In this section, we consider the extension of the results of Sect. 4.1.1 to the grand canonical ensemble, patterned after the transition from a quantum-mechanical system where the total particle number $N$ is fixed to a quantum field theoretical system where the total particle number $N$ varies due to the creation and the annihilation of particles.

In order to obtain the density matrix $\hat{\rho}_{\mathrm{GC}}(\beta)$ of the grand canonical ensemble, we replace the Hamiltonian operator $\hat{H}(\{\boldsymbol{q}_j, \boldsymbol{p}_j\})$ of the density matrix $\hat{\rho}_{\mathrm{C}}(\beta)$ of the canonical ensemble by the operator $\hat{H} - \mu\hat{N}$,

$$\hat{\rho}_{\mathrm{GC}}(\beta) = \exp\left[-\beta(\hat{H} - \mu\hat{N})\right], \tag{4.1.10}$$

where $\mu$ is the chemical potential, $\exp[\beta\mu]$ is the fugacity and $\hat{N}$ is the particle number operator defined by

$$\hat{N} \equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\psi_\alpha^\dagger(\tau, \boldsymbol{x})\psi_\alpha(\tau, \boldsymbol{x}).$$

Under the assumption of a two-body interaction $V_{\alpha\beta;\gamma\delta}(\boldsymbol{x} - \boldsymbol{y})$ of the system, we can accomplish our goal by replacing the Minkowskian action functional

$$I_{\mathrm{M}}[\{\boldsymbol{q}_j\}_{j=1}^N] \equiv \int_{t_i}^{t_f} \mathrm{d}t L_{\mathrm{M}}\left(\left\{\boldsymbol{q}_j(t), \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}_j(t)\right\}_{j=1}^N\right)$$

by the modified Minkowskian action functional

$$\begin{aligned}
&I'_{\mathrm{M}}[\psi_\alpha] \\
&\equiv \int \mathrm{d}^4x\{\mathcal{L}_{\mathrm{M}}(\psi_\alpha(x), \partial_\mu\psi_\alpha(x)) + \mu\psi_\alpha^\dagger(x)\psi_\alpha(x)\} \\
&\equiv \int \mathrm{d}^4x\mathcal{L}'_{\mathrm{M}}(\psi_\alpha(x), \partial_\mu\psi_\alpha(x)) \\
&= \int_{t_i}^{t_f} \mathrm{d}t\left\{\int \mathrm{d}^3\boldsymbol{x}\psi_\alpha^\dagger(t, \boldsymbol{x})\left(i\frac{\partial}{\partial t} + \mu + \frac{1}{2m}\nabla^2\right)\psi_\alpha(t, \boldsymbol{x})\right. \\
&\quad \left. -\frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\psi_\alpha^\dagger(t, \boldsymbol{x})\psi_\beta^\dagger(t, \boldsymbol{y})V_{\alpha\beta;\gamma\delta}(\boldsymbol{x} - \boldsymbol{y})\psi_\delta(t, \boldsymbol{y})\psi_\gamma(t, \boldsymbol{x})\right\},
\end{aligned} \tag{4.1.11a}$$

where the modified Minkowskian Lagrangian density is given by

$$\mathcal{L}'_{\mathrm{M}}(\psi_\alpha(x), \partial_\mu\psi_\alpha(x)) = \mathcal{L}_{\mathrm{M}}(\psi_\alpha(x), \partial_\mu\psi_\alpha(x)) + \mu\psi_\alpha^\dagger(x)\psi_\alpha(x), \tag{4.1.11b}$$

and by replacing the Euclidean action functional

$$I_{\mathrm{E}}[\{\boldsymbol{q}_j\}_{j=1}^N] = \int_0^\beta \mathrm{d}\tau L_{\mathrm{E}}\left(\left\{\boldsymbol{q}_j(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{q}_j(\tau)\right\}_{j=1}^N\right)$$

with the modified Euclidean action functional

$$
\begin{aligned}
&I'_{\mathrm{E}}[\psi_\alpha] \\
&\equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\{\mathcal{L}_{\mathrm{E}}(\psi_\alpha(\tau,\boldsymbol{x}),\partial_\mu\psi_\alpha(\tau,\boldsymbol{x})) + \mu\psi_\alpha^\dagger(\tau,\boldsymbol{x})\psi_\alpha(\tau,\boldsymbol{x})\} \\
&\equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\mathcal{L}'_{\mathrm{E}}(\psi_\alpha(x),\partial_\mu\psi_\alpha(x)) \\
&= \int_0^\beta \mathrm{d}\tau \left\{\int \mathrm{d}^3\boldsymbol{x}\psi_\alpha^\dagger(\tau,\boldsymbol{x})\left(-\frac{\partial}{\partial\tau} + \mu + \frac{1}{2m}\nabla^2\right)\psi_\alpha(\tau,\boldsymbol{x})\right. \\
&\quad \left. -\frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\psi_\alpha^\dagger(\tau,\boldsymbol{x})\psi_\beta^\dagger(\tau,\boldsymbol{y})V_{\alpha\beta;\gamma\delta}(\boldsymbol{x}-\boldsymbol{y})\psi_\delta(\tau,\boldsymbol{y})\psi_\gamma(\tau,\boldsymbol{x})\right\},
\end{aligned}
\tag{4.1.12a}
$$

where the modified Euclidean Lagrangian density is given by

$$
\begin{aligned}
&\mathcal{L}'_{\mathrm{E}}(\psi_\alpha(\tau,\boldsymbol{x}),\partial_\mu\psi_\alpha(\tau,\boldsymbol{x})) \\
&\quad = \mathcal{L}_{\mathrm{E}}(\psi_\alpha(\tau,\boldsymbol{x}),\partial_\mu\psi_\alpha(\tau,\boldsymbol{x})) + \mu\psi_\alpha^\dagger(\tau,\boldsymbol{x})\psi_\alpha(\tau,\boldsymbol{x})\,.
\end{aligned}
\tag{4.1.12b}
$$

From this, we obtain the partition function $Z_{\mathrm{GC}}(\beta)$ of the grand canonical ensemble as the analytic continuation of the vacuum-to-vacuum transition amplitude

$$\langle 0,\mathrm{out}|0,\mathrm{in}\rangle$$

of quantum field theory, i.e.,

$$
\begin{aligned}
&Z_{\mathrm{GC}}(\beta) \\
&\equiv \mathrm{Tr}\hat{\rho}_{\mathrm{GC}}(\beta) \\
&\equiv \mathrm{Tr}\exp[-\beta(\hat{H}-\mu\hat{N})] \\
&\equiv \int \mathcal{D}[\psi^\dagger]\mathcal{D}[\psi]\exp\left[\int_0^\beta \mathrm{d}\tau\left\{\int \mathrm{d}^3x\mathcal{L}'_{\mathrm{E}}(\psi_\alpha(\tau,\boldsymbol{x}),\partial_\mu\psi_\alpha(\tau,\boldsymbol{x}))\right\}\right] \\
&= \int \mathcal{D}[\psi^\dagger]\mathcal{D}[\psi]\exp\left[\int_0^\beta \mathrm{d}\tau\left\{\int \mathrm{d}^3\boldsymbol{x}\psi_\alpha^\dagger(\tau,\boldsymbol{x})\right.\right. \\
&\quad \times\left(-\frac{\partial}{\partial\tau} + \mu + \frac{1}{2m}\nabla^2\right)\psi_\alpha(\tau,\boldsymbol{x}) \\
&\quad \left.\left. -\frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\psi_\alpha^\dagger(\tau,\boldsymbol{x})\psi_\beta^\dagger(\tau,\boldsymbol{y})V_{\alpha\beta;\gamma\delta}(\boldsymbol{x}-\boldsymbol{y})\psi_\delta(\tau,\boldsymbol{y})\psi_\gamma(\tau,\boldsymbol{x})\right\}\right].
\end{aligned}
\tag{4.1.13}
$$

We can define the partition function $Z_{\mathrm{GC}}[\beta; J_\alpha, J_\beta^\dagger]$ of the grand canonical ensemble in the presence of the external hook $\{J_\alpha(\tau,\boldsymbol{x}), J_\beta^\dagger(\tau,\boldsymbol{x})\}$ as the analytic continuation of the generating functional of the "full" Green's functions

$Z[J]$, (2.1.30), by choosing

$$\mathcal{L}_{\mathrm{E}}^{\mathrm{ext}}(\tau,\boldsymbol{x}) \equiv \psi_\alpha^\dagger(\tau,\boldsymbol{x})J_\alpha(\tau,\boldsymbol{x}) + J_\beta^\dagger(\tau,\boldsymbol{x})\psi_\beta(\tau,\boldsymbol{x}). \tag{4.1.14}$$

Namely, we have

$$\begin{aligned} &Z_{\mathrm{GC}}[\beta; J_\alpha, J_\beta^\dagger] \\ &= C\int \mathcal{D}[\psi^\dagger]\mathcal{D}[\psi]\exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\{\mathcal{L}'_{\mathrm{E}}(\psi_\alpha(\tau,\boldsymbol{x}),\partial_\mu\psi_\alpha(\tau,\boldsymbol{x})) \right. \\ &\quad \left. +\mathcal{L}_{\mathrm{E}}^{\mathrm{ext}}(\tau,\boldsymbol{x})\}\right] \end{aligned} \tag{4.1.15}$$

$$\begin{aligned} &= C\exp\left[-\frac{1}{2}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\left\{\frac{\delta}{\delta J_\alpha(\tau,\boldsymbol{x})}\frac{\delta}{\delta J_\beta(\tau,\boldsymbol{y})}V_{\alpha\beta;\gamma\delta}(\boldsymbol{x}-\boldsymbol{y})\right.\right. \\ &\quad \left.\left.\times\frac{\delta}{\delta J_\delta^\dagger(\tau,\boldsymbol{y})}\frac{\delta}{\delta J_\gamma^\dagger(\tau,\boldsymbol{x})}\right\}\right] \\ &\quad \times\int \mathcal{D}[\psi^\dagger]\mathcal{D}[\psi]\exp\left[\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\right. \\ &\quad \times\left\{\psi_\alpha^\dagger(\tau,\boldsymbol{x})\left(-\frac{\partial}{\partial\tau}+\mu+\frac{1}{2m}\nabla^2\right)\psi_\alpha(\tau,\boldsymbol{x})\right. \\ &\quad \left.\left.+\psi_\alpha^\dagger(\tau,\boldsymbol{x})J_\alpha(\tau,\boldsymbol{x})+J_\alpha^\dagger(\tau,\boldsymbol{x})\psi_\alpha(\tau,\boldsymbol{x})\right\}\right] \\ &= C'\exp\left[-\frac{1}{2}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\left\{\frac{\delta}{\delta J_\alpha(\tau,\boldsymbol{x})}\frac{\delta}{\delta J_\beta(\tau,\boldsymbol{y})}V_{\alpha\beta;\gamma\delta}(\boldsymbol{x}-\boldsymbol{y})\right.\right. \\ &\quad \left.\left.\times\frac{\delta}{\delta J_\delta^\dagger(\tau,\boldsymbol{y})}\frac{\delta}{\delta J_\gamma^\dagger(\tau,\boldsymbol{x})}\right\}\right] \\ &\quad \times\exp\left[-\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\int_0^\beta \mathrm{d}\tau'\int \mathrm{d}^3\boldsymbol{y}J_\alpha^\dagger(\tau,\boldsymbol{x})G_0(\tau-\tau',\boldsymbol{x}-\boldsymbol{y})\right. \\ &\quad \left.\times J_\alpha(\tau',\boldsymbol{y})\right], \end{aligned} \tag{4.1.16}$$

where $G_0(\tau-\tau',\boldsymbol{x}-\boldsymbol{y})$ is the "free" temperature Green's function given by

$$\begin{aligned} &G_0(\tau-\tau',\boldsymbol{x}-\boldsymbol{y}) \\ &\quad = \left(-\frac{\partial}{\partial\tau}+\mu+\frac{1}{2m}\nabla^2\right)^{-1}\delta(\tau-\tau')\delta^3(\boldsymbol{x}-\boldsymbol{y}), \end{aligned} \tag{4.1.17}$$

and the normalization constant $C'$ is chosen such that

$$\begin{aligned}
C' &= Z_{\mathrm{GC}}[\beta; J_\alpha = J_\beta^\dagger = 0]|_{V=0} \\
&= Z_{\mathrm{GC}}(\beta; \text{free}) \\
&= \begin{cases} \prod_{\boldsymbol{k}} \left(1 - \exp\left[-\beta\left(\frac{\boldsymbol{k}^2}{2m} - \mu\right)\right]\right)^{-1}, & \text{B.E. statistics} \quad (4.1.18a) \\ \prod_{\boldsymbol{k}} \left(1 + \exp\left[-\beta\left(\frac{\boldsymbol{k}^2}{2m} - \mu\right)\right]\right), & \text{F.D. statistics.} \quad (4.1.18b) \end{cases}
\end{aligned}$$

Generally, when the interaction Lagrangian density is given by

$$\mathcal{L}_{\mathrm{E}}'^{\,\mathrm{int}}(\psi_\alpha^\dagger(\tau, \boldsymbol{x}), \psi_\beta(\tau, \boldsymbol{x}))\ , \tag{4.1.19a}$$

and the interaction action functional is given by

$$I_{\mathrm{E}}'^{\,\mathrm{int}}[\psi^\dagger, \psi] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \mathcal{L}_{\mathrm{E}}'^{\,\mathrm{int}}(\psi_\alpha^\dagger(\tau, \boldsymbol{x}), \psi_\beta(\tau, \boldsymbol{x})), \tag{4.1.19b}$$

the partition function $Z_{\mathrm{GC}}[\beta; J_\alpha, J_\beta^\dagger]$ of the grand canonical ensemble in the presence of the external hook is given by

$$\begin{aligned}
&\frac{Z_{\mathrm{GC}}[\beta; J_\alpha, J_\beta^\dagger]}{Z_0} \\
&= \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x} \mathcal{L}_{\mathrm{E}}'^{\,\mathrm{int}}\left(\frac{\delta}{\delta J_\alpha(\tau, \boldsymbol{x})}, \frac{\delta}{\delta J_\beta^\dagger(\tau, \boldsymbol{x})}\right)\right] \\
&\quad \times \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x} \int_0^\beta \mathrm{d}\tau' \int \mathrm{d}^3\boldsymbol{y} J_\alpha^\dagger(\tau, \boldsymbol{x}) G_0(\tau - \tau', \boldsymbol{x} - \boldsymbol{y}) \right. \\
&\quad \left. \times J_\alpha(\tau', \boldsymbol{y})\right],
\end{aligned} \tag{4.1.20}$$

where $Z_0$ is the appropriate normalization constant independent of $J_\alpha$ and $J_\beta^\dagger$. We obtain the diagrammatic expansion of $Z_{\mathrm{GC}}(\beta)$ in terms of $G_0$ and $V$ by the power series expansion of the interaction Lagrangian density

$$\mathcal{L}_{\mathrm{E}}'^{\,\mathrm{int}}\left(\frac{\delta}{\delta J_\alpha(\tau, \boldsymbol{x})}, \frac{\delta}{\delta J_\beta^\dagger(\tau, \boldsymbol{x})}\right)$$

and setting the external hooks equal to 0.

Next, we define the one-body "full" Green's function $G_{\alpha,\beta}(\tau, \boldsymbol{x}; \tau', \boldsymbol{x}')_J$ in the presence of the external hook by

$$
\begin{aligned}
&G_{\alpha,\beta}(\tau,\boldsymbol{x};\tau',\boldsymbol{x}')_J \\
&\equiv -\frac{1}{Z_{\mathrm{GC}}[\beta;J_\alpha,J_\beta^\dagger]}\frac{\delta}{\delta J_\beta(\tau',\boldsymbol{x}')}\frac{\delta}{\delta J_\alpha^\dagger(\tau,\boldsymbol{x})}Z_{\mathrm{GC}}[\beta;J_\alpha,J_\beta^\dagger] \\
&= \frac{\int\mathcal{D}[\psi^\dagger]\mathcal{D}[\psi]\psi_\alpha(\tau,\boldsymbol{x})\psi_\beta^\dagger(\tau',\boldsymbol{x}')\exp[I_{\mathrm{E}}'[\psi]+I_{\mathrm{E}}'^{\,\mathrm{int}}[\psi^\dagger,\psi]]}{\int\mathcal{D}[\psi^\dagger]\mathcal{D}[\psi]\exp[I_{\mathrm{E}}'[\psi]+I_{\mathrm{E}}'^{\,\mathrm{int}}[\psi^\dagger,\psi]]}\,.
\end{aligned}
\tag{4.1.21}
$$

By setting $J = J^\dagger = 0$ in (4.1.21), we obtain the *linked cluster expansion* of the one-body "full" Green's function in terms of $G_0$ and $V$,

$$
\begin{aligned}
&G_{\alpha,\beta}(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}') \\
&= -\frac{1}{Z_{\mathrm{GC}}[\beta;J,J^\dagger]}\frac{\delta}{\delta J_\beta(\tau',\boldsymbol{x}')}\frac{\delta}{\delta J_\alpha^\dagger(\tau,\boldsymbol{x})}Z_{\mathrm{GC}}[\beta;J,J^\dagger]|_{J=J^\dagger=0}\,.
\end{aligned}
\tag{4.1.22}
$$

Generally, we have the thermal expectation value of the $\mathrm{T}_\tau$-ordered product of the operators, $\{\hat{O}_j(\tau_j,\boldsymbol{x}_j)\}_{j=1}^n$, by analytic continuation of (2.1.29) as

$$
\begin{aligned}
&\langle \mathrm{T}_\tau\{\hat{O}_1(\tau_1,\boldsymbol{x}_1)\cdots\hat{O}_n(\tau_n,\boldsymbol{x}_n)\}\rangle_{\mathrm{GC}} \\
&= \mathrm{Tr}[\hat{\rho}_{\mathrm{GC}}(\beta)\mathrm{T}_\tau\{\hat{O}_1(\tau_1,\boldsymbol{x}_1)\cdots\hat{O}_n(\tau_n,\boldsymbol{x}_n)\}]/\mathrm{Tr}[\hat{\rho}_{\mathrm{GC}}(\beta)] \\
&= \frac{\int\mathcal{D}[\psi^\dagger]\mathcal{D}[\psi]O_1(\tau_1,\boldsymbol{x}_1)\cdots O_n(\tau_n,\boldsymbol{x}_n)\exp[I_{\mathrm{E}}'[\psi]]}{\int\mathcal{D}[\psi^\dagger]\mathcal{D}[\psi]\exp[I_{\mathrm{E}}'[\psi]]}\,.
\end{aligned}
\tag{4.1.23}
$$

In this section, we regard $\psi_\alpha(\tau,\boldsymbol{x})$ as a nonrelativistic complex scalar field. We can immediately extend to the case of a nonrelativistic fermion field as long as we pay attention to the order of the indices in $V_{\alpha\beta;\delta\gamma}(\boldsymbol{x}-\boldsymbol{y})$. For the *relativistic spinors* in Minkowskian space-time and Euclidean space-time, the reader is referred to Appendix 5.

In the discussion of Sect. 4.1, our argument is based on analytic continuation and the somewhat arbitrary introduction of the chemical potential $\mu$. In order to remedy these deficiencies, we move on to the Fradkin construction.

## 4.2 Fradkin Construction

In this section, we obtain the path integral representation of the partition function $Z_{\mathrm{GC}}(\beta)$ of the grand canonical ensemble based on the extended Matsubara representation of quantum field theory at finite temperature. The argument of this section validates the extremely heuristic argument of Sect. 4.1.

By the extended Matsubara representation, we mean that we use the operator $\hat{H}-\mu\hat{N}$ instead of the Hamiltonian operator $\hat{H}$ in the density matrix

$\hat{\rho}_{\mathrm{GC}}(\beta)$ of the grand canonical ensemble. In this respect, the interaction picture and the Heisenberg picture in this section are different from the standard ones.

In Sect. 4.2.1, through the introduction of the interaction picture, we obtain the $\hat{S}(\beta)$ operator of Dyson. Next, we introduce coupling with the external hook, and obtain the density matrix $\hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$ in the presence of the external hook. In the limit of

$$J=\bar{\eta}=\eta=0,$$

we can define the Heisenberg picture.

In Sect. 4.2.2, we derive the equation of motion of the partition function

$$Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])=\mathrm{Tr}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$$

of the grand canonical ensemble from the equation of motion of the field operator and the equal "time" canonical (anti-)commutators. By a similar method to that used in Sect. 2.3, we obtain the path integral representation of the partition functions, $Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$ and $Z_{\mathrm{GC}}(\beta)$, of an interacting boson–fermion system (the Fradkin construction). Lastly, we derive the (anti-) periodicity of the boson (fermion) "full" temperature Green's function with period $\beta$, and we perform Fourier transformations of the boson field and the fermion field.

### 4.2.1 Density Matrix of Relativistic Quantum Field Theory at Finite Temperature

We consider the grand canonical ensemble of an interacting system of fermions (mass $m$) and bosons (mass $\kappa$) with Euclidean Lagrangian density in contact with a particle source $\mu$;

$$\begin{aligned}
&\mathcal{L}_{\mathrm{E}}'(\psi_{\mathrm{E}\alpha}(\tau,\boldsymbol{x}),\partial_\mu\psi_{\mathrm{E}\alpha}(\tau,\boldsymbol{x}),\phi(\tau,\boldsymbol{x}),\partial_\mu\phi(\tau,\boldsymbol{x}))\\
&=\bar{\psi}_{\mathrm{E}\ \alpha}(\tau,\boldsymbol{x})\{\mathrm{i}\gamma^k\partial_k+\mathrm{i}\gamma^4\left(\frac{\partial}{\partial\tau}-\mu\right)-m\}_{\alpha,\beta}\psi_{\mathrm{E}\ \beta}(\tau,\boldsymbol{x})\\
&\quad+\frac{1}{2}\phi(\tau,\boldsymbol{x})\left(\frac{\partial^2}{\partial x_\nu^2}-\kappa^2\right)\phi(\tau,\boldsymbol{x})\\
&\quad+\frac{1}{2}g\mathrm{Tr}\{\gamma[\bar{\psi}_{\mathrm{E}}(\tau,\boldsymbol{x}),\psi_{\mathrm{E}}(\tau,\boldsymbol{x})]\}\phi(\tau,\boldsymbol{x})\ .
\end{aligned}\tag{4.2.1}$$

The density matrix $\hat{\rho}_{\mathrm{GC}}(\beta)$ of the grand canonical ensemble in the Schrödinger picture is given by

$$\hat{\rho}_{\mathrm{GC}}(\beta)=\exp[-\beta(\hat{H}-\mu\hat{N})],\qquad \beta=\frac{1}{k_{\mathrm{B}}T},\tag{4.2.2}$$

where the total Hamiltonian $\hat{H}$ is split into two parts:

$$\hat{H}_0 = \text{ free Hamiltonian for fermion (mass } m\text{) and boson (mass } \kappa\text{).}$$

$$\hat{H}_1 = -\int d^3\boldsymbol{x}\hat{\jmath}(\boldsymbol{x})\hat{\phi}(\boldsymbol{x}), \tag{4.2.3}$$

with the "current" given by

$$\hat{\jmath}(\boldsymbol{x}) = \frac{1}{2}g\mathrm{Tr}\{\gamma[\hat{\psi}_{\mathrm{E}}^{\dagger}(\boldsymbol{x})\gamma^4), \hat{\psi}_{\mathrm{E}}(\boldsymbol{x})]\}, \tag{4.2.4}$$

and

$$\hat{N} = \frac{1}{2}\int d^3\boldsymbol{x}\mathrm{Tr}\{-\gamma^4[(\hat{\psi}_{\mathrm{E}}^{\dagger}(\boldsymbol{x})\gamma^4), \hat{\psi}_{\mathrm{E}}(\boldsymbol{x})]\}. \tag{4.2.5}$$

By the standard method of quantum field theory, we use the interaction picture with $\hat{N}$ included in the free part, and obtain

$$\hat{\rho}_{\mathrm{GC}}(\beta) = \hat{\rho}_0(\beta)\hat{S}(\beta), \tag{4.2.6a}$$

$$\hat{\rho}_0(\beta) = \exp\left[-\beta(\hat{H}_0 - \mu\hat{N})\right], \tag{4.2.7}$$

$$\hat{S}(\beta) = \mathrm{T}_\tau\left\{\exp\left[-\int_0^\beta d\tau\int d^3\boldsymbol{x}\hat{\mathcal{H}}_1(\tau,\boldsymbol{x})\right]\right\} \tag{4.2.8a}$$

and

$$\hat{\mathcal{H}}_1(\tau,\boldsymbol{x}) = -\hat{\jmath}^{(\mathrm{I})}(\tau,\boldsymbol{x})\hat{\phi}^{(\mathrm{I})}(\tau,\boldsymbol{x}). \tag{4.2.9}$$

We know that the interaction picture operator $\hat{f}^{(\mathrm{I})}(\tau,\boldsymbol{x})$ is related to the Schrödinger picture operator $\hat{f}(\boldsymbol{x})$ through

$$\hat{f}^{(\mathrm{I})}(\tau,\boldsymbol{x}) = \hat{\rho}_0^{-1}(\tau)\cdot\hat{f}(\boldsymbol{x})\cdot\hat{\rho}_0(\tau). \tag{4.2.10}$$

We introduce the external hook $\{J(\tau,\boldsymbol{x}), \bar{\eta}_\alpha(\tau,\boldsymbol{x}), \eta_\beta(\tau,\boldsymbol{x})\}$ in the interaction picture, and obtain

$$\begin{aligned}&\hat{\mathcal{H}}_1^{\mathrm{int}}(\tau,\boldsymbol{x})\\&= -\{[\hat{\jmath}^{(\mathrm{I})}(\tau,\boldsymbol{x}) + J(\tau,\boldsymbol{x})]\hat{\phi}^{(\mathrm{I})}(\tau,\boldsymbol{x})\\&\quad+\bar{\eta}_\alpha(\tau,\boldsymbol{x})\hat{\psi}_{\mathrm{E}\,\alpha}^{(\mathrm{I})}(\tau,\boldsymbol{x}) + (\hat{\psi}_{\mathrm{E}}^{\dagger(\mathrm{I})}(\tau,\boldsymbol{x})\gamma^4)_\beta\eta_\beta(\tau,\boldsymbol{x}))\}.\end{aligned} \tag{4.2.11}$$

We replace (4.2.6a), (4.2.7) and (4.2.8a) with

$$\hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta]) = \hat{\rho}_0(\beta)\hat{S}(\beta;[J,\bar{\eta},\eta]), \tag{4.2.6b}$$

$$\hat{\rho}_0(\beta) = \exp\left[-\beta(\hat{H}_0 - \mu\hat{N})\right], \tag{4.2.7}$$

$$\hat{S}(\beta;[J,\bar{\eta},\eta]) = \mathrm{T}_\tau\left\{\exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\hat{\mathcal{H}}_1^{\mathrm{int}}(\tau,\boldsymbol{x})\right]\right\}. \tag{4.2.8b}$$

Here, we have

(a) $0 \le \tau \le \beta$.

$$\begin{aligned}
&\frac{\delta}{\delta J(\tau,\boldsymbol{x})}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])|_{J=\bar{\eta}=\eta=0} \\
&= \hat{\rho}_0(\beta)\mathrm{T}_\tau\left\{\hat{\phi}^{(\mathrm{I})}(\tau,\boldsymbol{x})\exp\left[-\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\right]\hat{\mathcal{H}}_1^{\mathrm{int}}(\tau,\boldsymbol{x})]\right\}|_{J=\bar{\eta}=\eta=0} \\
&= \hat{\rho}_0(\beta)\mathrm{T}_\tau\left\{\exp\left[-\int_\tau^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\hat{\mathcal{H}}_1^{\mathrm{int}}\right]\right\}\hat{\phi}^{(\mathrm{I})}(\tau,\boldsymbol{x}) \\
&\quad\times\mathrm{T}_\tau\left\{\exp\left[-\int_0^\tau \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\hat{\mathcal{H}}_1^{\mathrm{int}}\right]\right\}|_{J=\bar{\eta}=\eta=0} \\
&= \hat{\rho}_0(\beta)\hat{S}(\beta;[J,\bar{\eta},\eta])\hat{S}(-\tau;[J,\bar{\eta},\eta])\hat{\phi}^{(\mathrm{I})}(\tau,\boldsymbol{x})\hat{S}(\tau;[J,\bar{\eta},\eta])|_{J=\bar{\eta}=\eta=0} \\
&= \hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])\{\hat{\rho}_0(\tau)\hat{S}(\tau;[J,\bar{\eta},\eta])\}^{-1} \\
&\quad\times\hat{\phi}(\boldsymbol{x})\{\hat{\rho}_0(\tau)\hat{S}(\tau;[J,\bar{\eta},\eta])\}|_{J=\bar{\eta}=\eta=0} \\
&= \hat{\rho}_{\mathrm{GC}}(\beta)\hat{\phi}(\tau,\boldsymbol{x}).
\end{aligned}$$

Thus, we obtain

$$\frac{\delta}{\delta J(\tau,\boldsymbol{x})}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])|_{J=\bar{\eta}=\eta=0} = \hat{\rho}_{\mathrm{GC}}(\beta)\hat{\phi}(\tau,\boldsymbol{x}). \tag{4.2.12}$$

Likewise, we obtain

$$\frac{\delta}{\delta\bar{\eta}_\alpha(\tau,\boldsymbol{x})}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])|_{J=\bar{\eta}=\eta=0} = \hat{\rho}_{\mathrm{GC}}(\beta)\hat{\psi}_{\mathrm{E}\ \alpha}(\tau,\boldsymbol{x}), \tag{4.2.13}$$

$$\frac{\delta}{\delta\eta_\beta(\tau,\boldsymbol{x})}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])|_{J=\bar{\eta}=\eta=0} = -\hat{\rho}_{\mathrm{GC}}(\beta)(\hat{\psi}_{\mathrm{E}}^\dagger(\tau,\boldsymbol{x})\gamma^4)_\beta, \tag{4.2.14}$$

and

$$\begin{aligned}
&\frac{\delta^2}{\delta\bar{\eta}_\alpha(\tau,\boldsymbol{x})\delta\eta_\beta(\tau',\boldsymbol{x})}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])|_{J=\bar{\eta}=\eta=0} \\
&= -\hat{\rho}_{\mathrm{GC}}(\beta)\mathrm{T}_\tau\{\hat{\psi}_{\mathrm{E}\ \alpha}(\tau,\boldsymbol{x})(\hat{\psi}_{\mathrm{E}}^\dagger\ (\tau',\boldsymbol{x}')\gamma^4)_\beta\}\ .
\end{aligned} \tag{4.2.15}$$

The Heisenberg picture operator $\hat{f}(\tau, \boldsymbol{x})$ is related to the Schrödinger picture operator $\hat{f}(\boldsymbol{x})$ by

$$\hat{f}(\tau, \boldsymbol{x}) = \hat{\rho}_{\mathrm{GC}}^{-1}(\tau) \cdot \hat{f}(\boldsymbol{x}) \cdot \hat{\rho}_{\mathrm{GC}}(\tau). \tag{4.2.16}$$

(b) $\tau \notin [0, \beta]$. As for $\tau \notin [0, \beta]$, the functional derivatives of $\hat{\rho}_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta])$ with respect to $\{J, \bar{\eta}, \eta\}$ all vanish.

### 4.2.2 Functional Differential Equation of the Partition Function of the Grand Canonical Ensemble

We use the "equation of motion" of $\hat{\psi}_{\mathrm{E}\ \alpha}(\tau, \boldsymbol{x})$ and $\hat{\phi}(\tau, \boldsymbol{x})$,

$$\left\{ \mathrm{i}\gamma^k \partial_k + \mathrm{i}\gamma^4 \left( \frac{\partial}{\partial \tau} - \mu \right) - m + g\gamma\hat{\phi}(\tau, \boldsymbol{x}) \right\}_{\beta,\alpha} \hat{\psi}_{\mathrm{E}\ \alpha}(\tau, \boldsymbol{x}) = 0, \tag{4.2.17}$$

$$(\hat{\psi}_{\mathrm{E}}^{\dagger}(\tau, \boldsymbol{x})\gamma^4)_{\beta} \left\{ \mathrm{i}\gamma^k \partial_k + \mathrm{i}\gamma^4 \left( \frac{\partial}{\partial \tau} - \mu \right) - m + g\gamma\hat{\phi}(\tau, \boldsymbol{x}) \right\}_{\beta,\alpha}^{\mathrm{T}} = 0, \tag{4.2.18}$$

$$\left( \frac{\partial^2}{\partial x_\nu^2} - \kappa^2 \right) \hat{\phi}(\tau, \boldsymbol{x}) + g\mathrm{Tr}\{\gamma(\hat{\psi}_{\mathrm{E}}^{\dagger}(\tau, \boldsymbol{x})\gamma^4)\hat{\psi}_{\mathrm{E}}(\tau, \boldsymbol{x})\} = 0, \tag{4.2.19}$$

and the equal "time" canonical (anti-)commuters

$$\begin{aligned} &\delta(\tau - \tau')\{\hat{\psi}_{\mathrm{E}\ \alpha}(\tau, \boldsymbol{x}), (\hat{\psi}_{\mathrm{E}}^{\dagger}(\tau', \boldsymbol{x}')\gamma^4)_{\beta}\} \\ &\quad = \delta_{\alpha\beta}\delta(\tau - \tau')\delta(\boldsymbol{x} - \boldsymbol{x}'), \end{aligned} \tag{4.2.20a}$$

$$\delta(\tau - \tau') \left[ \hat{\phi}(\tau, \boldsymbol{x}), \frac{\partial}{\partial \tau'}\hat{\phi}(\tau', \boldsymbol{x}') \right] = \delta(\tau - \tau')\delta^3(\boldsymbol{x} - \boldsymbol{y}): \tag{4.2.20b}$$

all remaining equal "time" (anti-)commutators = 0. (4.2.20c)

We obtain the equations of motion of the partition function $Z_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta])$ of the grand canonical ensemble

$$Z_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta]) = \mathrm{Tr}\hat{\rho}_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta]) \tag{4.2.21}$$

in the presence of the external hook $\{J, \bar{\eta}, \eta\}$ from (4.2.12), (4.2.13) and (4.2.14) as

$$\left\{ \mathrm{i}\gamma^k\partial_k + \mathrm{i}\gamma^4\left(\frac{\partial}{\partial\tau} - \mu\right) - m + g\gamma\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(\tau,\boldsymbol{x})}\right\}_{\beta,\alpha}$$
$$\times\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_\alpha(\tau,\bar{x})}Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$$
$$= -\eta_\beta(\tau,\boldsymbol{x})Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta]) \ , \qquad (4.2.22)$$

$$\left\{ \mathrm{i}\gamma^k\partial_k + \mathrm{i}\gamma^4\left(\frac{\partial}{\partial\tau} + \mu\right) + m - g\gamma\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(\tau,\boldsymbol{x})}\right\}^{\mathrm{T}}_{\beta,\alpha}$$
$$\times\mathrm{i}\frac{\delta}{\delta\eta_\beta(\tau,\boldsymbol{x})}Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$$
$$= \bar{\eta}_\alpha(\tau,\boldsymbol{x})Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta]), \qquad (4.2.23)$$

$$\left\{\left(\frac{\partial^2}{\partial x_\nu^2} - \kappa^2\right)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(\tau,\boldsymbol{x})} - g\gamma_{\beta\alpha}\frac{\delta^2}{\delta\bar{\eta}_\alpha(\tau,\boldsymbol{x})\delta\eta_\beta(\tau,\boldsymbol{x})}\right\}$$
$$\times Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$$
$$= J(\tau,\boldsymbol{x})Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta]). \qquad (4.2.24)$$

We can solve the functional differential equations, (4.2.22), (4.2.23) and (4.2.24), by the method of Sect. 2.3. As in Sect. 2.3, we define the functional Fourier transform $\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\bar{\psi}_{\mathrm{E}}])$ of $Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$ by

$$Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$$
$$\equiv \int \mathcal{D}[\bar{\psi}_{\mathrm{E}}]\mathcal{D}[\psi_{\mathrm{E}}]\mathcal{D}[\phi]\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\bar{\psi}_{\mathrm{E}}])$$
$$\times\exp\left[\mathrm{i}\int_0^\beta \mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}\right.$$
$$\left.\times\{J(\tau,\boldsymbol{x})\phi(\tau,\boldsymbol{x}) + \bar{\eta}_\alpha(\tau,\boldsymbol{x})\psi_{\mathrm{E}\ \alpha}(\tau,\boldsymbol{x}) + \bar{\psi}_{\mathrm{E}\ \beta}(\tau,\boldsymbol{x})\eta_\beta(\tau,\boldsymbol{x})\}\right].$$

We obtain functional differential equations satisfied by the functional Fourier transform $\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\bar{\psi}_{\mathrm{E}}])$ from (4.2.22), (4.2.23) and (4.2.24), after functional integration by parts on the right-hand sides involving $\bar{\eta}_\alpha$, $\eta_\beta$ and $J$ as

$$\frac{\delta}{\delta\psi_{\mathrm{E}\ \alpha}(\tau,\boldsymbol{x})}\ln\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\bar{\psi}_{\mathrm{E}}])$$
$$= \frac{\delta}{\delta\psi_{\mathrm{E}\ \alpha}(\tau,\boldsymbol{x})}\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}\mathcal{L}'_{\mathrm{E}}((4.2.1)),$$

$$\frac{\delta}{\delta\bar{\psi}_{\mathrm{E}\;\beta}(\tau,\boldsymbol{x})}\ln\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\bar{\psi}_{\mathrm{E}}])$$
$$=\frac{\delta}{\delta\bar{\psi}_{\mathrm{E}\;\beta}(\tau,\boldsymbol{x})}\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}\mathcal{L}'_{\mathrm{E}}((4.2.1)),$$

$$\frac{\delta}{\delta\phi(\tau,\boldsymbol{x})}\ln\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\bar{\psi}_{\mathrm{E}}])$$
$$=\frac{\delta}{\delta\phi(\tau,\boldsymbol{x})}\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}\mathcal{L}'_{\mathrm{E}}((4.2.1)),$$

which we can immediately integrate to obtain

$$\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\bar{\psi}_{\mathrm{E}}])=C\exp\left[\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}\mathcal{L}'_{\mathrm{E}}((4.2.1))\right].\tag{4.2.25a}$$

Thus, we have the path integral representation of $Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$ as

$$\begin{aligned}
&Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])\\
&=C\int\mathcal{D}[\bar{\psi}_{\mathrm{E}}]\mathcal{D}[\psi_{\mathrm{E}}]\mathcal{D}[\phi]\exp\left[\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}\{\mathcal{L}'_{\mathrm{E}}((4.2.1))\right.\\
&\qquad\left.+\mathrm{i}J(\tau,\boldsymbol{x})\phi(\tau,\boldsymbol{x})+\mathrm{i}\bar{\eta}_\alpha(\tau,\boldsymbol{x})\psi_{\mathrm{E}\;\alpha}(\tau,\boldsymbol{x})+\mathrm{i}\bar{\psi}_{\mathrm{E}\;\beta}(\tau,\boldsymbol{x})\eta_\beta(\tau,\boldsymbol{x})\}\right]\\
&=Z_0\exp\left[-g\gamma_{\beta\alpha}\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}\mathrm{i}\frac{\delta}{\delta\eta_\beta(\tau,\boldsymbol{x})}\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_\alpha(\tau,\boldsymbol{x})}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(\tau,\boldsymbol{x})}\right]\\
&\qquad\times\exp\left[\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\boldsymbol{x}'\right.\\
&\qquad\times\{-\tfrac{1}{2}J(\tau,\boldsymbol{x})D_0(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')J(\tau',\boldsymbol{x}')\\
&\qquad\left.+\bar{\eta}_\alpha(\tau,\boldsymbol{x})G_{\substack{0\\ \alpha\beta}}(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')\eta_\beta(\tau',\boldsymbol{x}')\}\right].
\end{aligned}\tag{4.2.25b}$$

The normalization constant $Z_0$ is chosen so that

$$\begin{aligned}
Z_0&=Z_{\mathrm{GC}}(\beta;J=\bar{\eta}=\eta=0,g=0)\\
&=\prod_{|\boldsymbol{p}|,|\boldsymbol{k}|}\{1+\exp[-\beta(\varepsilon_{\boldsymbol{p}}-\mu)]\}\\
&\qquad\times\{1+\exp[-\beta(\varepsilon_{\boldsymbol{p}}+\mu)]\}\{1-\exp[-\beta\omega_{\boldsymbol{k}}]\}^{-1},
\end{aligned}\tag{4.2.26}$$

with

$$\varepsilon_{\boldsymbol{p}} = (\boldsymbol{p}^2 + m^2)^{1/2}, \qquad \omega_{\boldsymbol{k}} = (\boldsymbol{k}^2 + \kappa^2)^{1/2}. \tag{4.2.27}$$

$D_0(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}')$ and $G_{\alpha\beta}^{0}(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}')$ are the "free" temperature Green's functions of the Bose field and the Fermi field, respectively, and are given by

$$\begin{aligned} &D_0(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}') \\ &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega_{\boldsymbol{k}}} \{(f_{\boldsymbol{k}} + 1)\exp[\mathrm{i}\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}') - \omega_{\boldsymbol{k}}(\tau - \tau')] \\ &\quad + f_{\boldsymbol{k}} \exp[-\mathrm{i}\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}') + \omega_{\boldsymbol{k}}(\tau - \tau')]\} , \end{aligned} \tag{4.2.28}$$

$$\begin{aligned} &G_{\alpha\beta}^{0}(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}') \\ &= (\mathrm{i}\gamma^{\nu}\partial_{\nu} + m)_{\alpha,\beta} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\varepsilon_{\boldsymbol{k}}} \\ &\quad \times \begin{cases} \{(N_{\boldsymbol{k}}^{+} - 1)\exp[\mathrm{i}\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}') - (\varepsilon_{\boldsymbol{k}} - \mu)(\tau - \tau')] \\ \;+ N_{\boldsymbol{k}}^{-} \exp[-\mathrm{i}\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}') + (\varepsilon_{\boldsymbol{k}} + \mu)(\tau - \tau')]\}, \\ \qquad\qquad\qquad\qquad \text{for } \tau > \tau', \\ \{N_{\boldsymbol{k}}^{+} \exp[-\mathrm{i}\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}') - (\varepsilon_{\boldsymbol{k}} - \mu)(\tau - \tau')] \\ \;+ (N_{\boldsymbol{k}}^{-} - 1)\exp[\mathrm{i}\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}') + (\varepsilon_{\boldsymbol{k}} + \mu)(\tau - \tau')]\}, \\ \qquad\qquad\qquad\qquad \text{for } \tau < \tau', \end{cases} \end{aligned} \tag{4.2.29}$$

where

$$\partial_4 \equiv \frac{\partial}{\partial \tau} - \mu \;, \; f_{\boldsymbol{k}} = \frac{1}{\exp[\beta\omega_{\boldsymbol{k}}] - 1} \;, \; N_{\boldsymbol{k}}^{\pm} = \frac{1}{\exp[\beta(\varepsilon_{\boldsymbol{k}} \mp \mu)] + 1}. \tag{4.2.30}$$

The $f_{\boldsymbol{k}}$ are the density of states at energy $\omega_{\boldsymbol{k}}$ of the Bose particles, and the $N_{\boldsymbol{k}}^{\pm}$ are the density of states at energy $\varepsilon_{\boldsymbol{k}}$ of the (anti-)Fermi particles.

For the path integral representation of $Z_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta])$, (4.2.25b), we have two ways of expressing it, just like the example in Sect. 2.3.3.

$$
\begin{aligned}
&Z_{\mathrm{GC}}(\beta;[J,\bar\eta,\eta])\\
&=Z_0\exp\left[-\frac{1}{2}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\int_0^\beta \mathrm{d}\tau'\int \mathrm{d}^3\boldsymbol{x}'\right.\\
&\quad\times\left\{J(\tau,\boldsymbol{x})-g\gamma_{\beta\alpha}\mathrm{i}\frac{\delta}{\delta\eta_\beta(\tau,\boldsymbol{x})}\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar\eta_\alpha(\tau,\boldsymbol{x})}\right\}\\
&\quad\left.\times D_0(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')\left\{J(\tau',\boldsymbol{x}')-g\gamma_{\beta\alpha}\mathrm{i}\frac{\delta}{\delta\eta_\beta(\tau',\boldsymbol{x}')}\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar\eta_\alpha(\tau',\boldsymbol{x}')}\right\}\right]\\
&\quad\times\exp\left[\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\int_0^\beta \mathrm{d}\tau'\int \mathrm{d}^3\boldsymbol{x}'\bar\eta_\alpha(\tau,\boldsymbol{x})G^{0}_{\alpha\beta}\right.\\
&\quad\left.\times(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')\eta_\beta(\tau',\boldsymbol{x}')\right] \qquad (4.2.31)
\end{aligned}
$$

$$
\begin{aligned}
&=Z_0\left\{\mathrm{Det}\left(1+gG_0(\tau,\boldsymbol{x})\gamma\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(\tau,\boldsymbol{x})}\right)\right\}^{-1} \qquad (4.2.32)\\
&\quad\times\exp\left[\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\int_0^\beta \mathrm{d}\tau'\int \mathrm{d}^3\boldsymbol{x}'\right.\\
&\quad\left.\times\bar\eta_\alpha(\tau,\boldsymbol{x})\left(1+gG_0(\tau,\boldsymbol{x})\gamma\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(\tau,\boldsymbol{x})}\right)^{-1}_{\alpha\varepsilon}G^{0}_{\varepsilon\beta}(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')\eta_\beta(\tau',\boldsymbol{x}')\right]\\
&\quad\times\exp\left[-\frac{1}{2}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\int_0^\beta \mathrm{d}\tau'\int \mathrm{d}^3\boldsymbol{x}'J(\tau,\boldsymbol{x})D_0(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')J(\tau',\boldsymbol{x}')\right].
\end{aligned}
$$

The thermal expectation value of the $\tau$-ordered function

$$f_{\tau\text{-ordered}}(\hat\psi,\hat\psi^\dagger\gamma^4,\hat\phi)$$

in the grand canonical ensemble is given by

$$
\begin{aligned}
&\langle f_{\tau\text{-ordered}}(\hat\psi,\hat\psi^\dagger\gamma^4,\hat\phi)\rangle\\
&\equiv\frac{\mathrm{Tr}\{\hat\rho_{\mathrm{GC}}(\beta)f_{\tau\text{-ordered}}(\hat\psi,\hat\psi^\dagger\gamma^4,\hat\phi)\}}{\mathrm{Tr}\hat\rho_{\mathrm{GC}}(\beta)}\\
&\equiv\frac{1}{Z_{\mathrm{GC}}(\beta;[J,\bar\eta,\eta])}f\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar\eta},\mathrm{i}\frac{\delta}{\delta\eta},\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)Z_{\mathrm{GC}}(\beta;[J,\bar\eta,\eta])|_{J=\bar\eta=\eta=0}.
\end{aligned}
\qquad (4.2.33)
$$

According to this formula, the one-body "full" temperature Green's functions of the Bose field and the Fermi field, $D(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')$ and $G_{\alpha\beta}(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}')$, are respectively given by

$$
\begin{aligned}
&D(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}') \\
&=\frac{\mathrm{Tr}\{\hat{\rho}_{\mathrm{GC}}(\beta)\mathrm{T}_\tau(\hat{\phi}(\tau,\boldsymbol{x})\hat{\phi}(\tau',\boldsymbol{x}'))\}}{\mathrm{Tr}\hat{\rho}_{\mathrm{GC}}(\beta)} \\
&=-\frac{1}{Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(\tau,\boldsymbol{x})}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(\tau',\boldsymbol{x}')}Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])|_{J=\bar{\eta}=\eta=0},
\end{aligned}
\tag{4.2.34}
$$

$$
\begin{aligned}
G_{\alpha\beta}(\tau-\tau',\boldsymbol{x}-\boldsymbol{x}') &= \frac{\mathrm{Tr}\{\hat{\rho}_{\mathrm{GC}}(\beta)\hat{\psi}_\alpha(\tau,\boldsymbol{x})(\hat{\psi}^\dagger(\tau',\boldsymbol{x}')\gamma^4)_\beta\}}{\mathrm{Tr}\hat{\rho}_{\mathrm{GC}}(\beta)} \\
&=-\frac{1}{Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])}\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}_\alpha(\tau,\boldsymbol{x})}\mathrm{i}\frac{\delta}{\delta\eta_\beta(\tau',\boldsymbol{x}')}Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])|_{J=\bar{\eta}=\eta=0}.
\end{aligned}
\tag{4.2.35}
$$

From the cyclicity of Tr and the (anti-)commutativity of $\hat{\phi}(\tau,\boldsymbol{x})$ ($\hat{\psi}_\alpha(\tau,\boldsymbol{x})$) under the $\mathrm{T}_\tau$-ordering symbol, we have

$$
D(\tau-\tau'<0,\boldsymbol{x}-\boldsymbol{x}') = +D(\tau-\tau'+\beta,\boldsymbol{x}-\boldsymbol{x}'), \tag{4.2.36}
$$

and

$$
G_{\alpha\beta}(\tau-\tau'<0,\boldsymbol{x}-\boldsymbol{x}') = -G_{\alpha\beta}(\tau-\tau'+\beta,\boldsymbol{x}-\boldsymbol{x}'), \tag{4.2.37}
$$

where

$$
0\le\tau,\tau'\le\beta,
$$

i.e., the boson (fermion) “full” temperature Green’s function is (anti-)periodic with the period $\beta$. From this, we have the Fourier decompositions as

$$
\begin{aligned}
&\hat{\phi}(\tau,\boldsymbol{x}) \\
&=\frac{1}{\beta}\sum_n\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega_{\boldsymbol{k}}}\{\exp[\mathrm{i}(\boldsymbol{kx}-\omega_n\tau)]a(\omega_n,\boldsymbol{k}) \\
&\quad+\exp[-\mathrm{i}(\boldsymbol{kx}-\omega_n\tau)]a^\dagger(\omega_n,\boldsymbol{k})\}\ ,
\end{aligned}
\tag{4.2.38}
$$

$$
\omega_n=\frac{2n\pi}{\beta},\qquad n=\text{integer},
$$

$$
\begin{aligned}
\hat{\psi}_\alpha(\tau,\boldsymbol{x}) = \ &\frac{1}{\beta}\sum_n\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\varepsilon_{\boldsymbol{k}}}\{\exp[\mathrm{i}(\boldsymbol{kx}-\omega_n\tau)]u_{n\alpha}(\boldsymbol{k})b(\omega_n,\boldsymbol{k}) \\
&+\exp[-\mathrm{i}(\boldsymbol{kx}-\omega_n\tau)]\bar{v}_{n\alpha}(\boldsymbol{k})\mathrm{d}^\dagger(\omega_n,\boldsymbol{k})\}\ ,
\end{aligned}
\tag{4.2.39}
$$

$$\omega_n = \frac{(2n+1)\pi}{\beta}, \qquad n = \text{integer},$$

where

$$[a(\omega_n, \boldsymbol{k}), a^\dagger(\omega_{n'}, \boldsymbol{k}')] = 2\omega_{\boldsymbol{k}}(2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')\delta_{n,n'}, \tag{4.2.40a}$$

$$[a(\omega_n, \boldsymbol{k}), a(\omega_{n'}, \boldsymbol{k}')] = [a^\dagger(\omega_n, \boldsymbol{k}), a^\dagger(\omega_{n'}, \boldsymbol{k}')] = 0\,, \tag{4.2.40b}$$

$$\begin{aligned}\{b(\omega_n, \boldsymbol{k}), b^\dagger(\omega_{n'}, \boldsymbol{k}')\} &= \{\mathrm{d}(\omega_n, \boldsymbol{k}), \mathrm{d}^\dagger(\omega_{n'}, \boldsymbol{k}')\} \\ &= 2\varepsilon_{\boldsymbol{k}}(2\pi)^3\delta^3(\boldsymbol{k} - \boldsymbol{k}')\delta_{n,n'}:\end{aligned} \tag{4.2.40c}$$

$$\text{the remaining anticommutators} = 0\,. \tag{4.2.40d}$$

## 4.3 Schwinger–Dyson Equation

In this section, we discuss the Schwinger–Dyson equation satisfied by the "full" temperature Green's functions, $D^J(x, y)$ and $G^J_{\alpha,\beta}(x, y)$, of the boson field and the fermion field in the presence of the external bosonic hook, $\{J(\tau, \boldsymbol{x})\}$.

In Sect. 4.3.1, we derive the Schwinger–Dyson equation satisfied by

$$D^J(x, y) \text{ and } G^J_{\alpha,\beta}(x, y)$$

from the equation of motion satisfied by the partition function

$$Z_{\mathrm{GC}}(\beta; [J, \bar{\eta}, \eta])$$

of the grand canonical ensemble in the presence of the hook,

$$\{J(\tau, \boldsymbol{x}), \bar{\eta}_\alpha(\tau, \boldsymbol{x}), \eta_\beta(\tau, \boldsymbol{x})\},$$

in the same manner as in Sect. 2.4, after setting the fermionic hooks equal to 0,

$$\bar{\eta}_\alpha(\tau, \boldsymbol{x}) = \eta_\beta(\tau, \boldsymbol{x}) = 0\,.$$

Next, we set

$$J(\tau, \boldsymbol{x}) = 0\,,$$

recovering the translation invariance of the system, and obtain the Schwinger–Dyson equation in momentum space as a result of the Fourier decomposition.

In Sect. 4.3.2, we consider the nonrelativistic limit of the fermion "free" temperature Green's function, and confirm the agreement with the result of Sect. 4.1.

### 4.3.1 The Schwinger–Dyson Equation

We define the one-body boson and fermion "full" temperature Green's functions, $D^J(x,y)$ and $G^J_{\alpha,\beta}(x,y)$, by

$$\begin{aligned} D^J(x,y) &\equiv -\langle \mathrm{T}_\tau(\hat{\phi}(x)\hat{\phi}(y))\rangle^J\Big|_{\bar{\eta}=\eta=0} \\ &\equiv -\frac{\delta^2}{\delta J(x)\delta J(y)} \ln Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])\Big|_{\bar{\eta}=\eta=0} \\ &\equiv -\frac{\delta}{\delta J(x)}\langle\hat{\phi}(y)\rangle^J\Big|_{\bar{\eta}=\eta=0}, \end{aligned} \tag{4.3.1}$$

and

$$\begin{aligned} &G^J_{\alpha,\beta}(x,y) \\ &\equiv +\langle \mathrm{T}_\tau(\hat{\psi}_\alpha(x)(\hat{\psi}^\dagger(y)\gamma^4)_\beta)\rangle^J\Big|_{\bar{\eta}=\eta=0} \\ &\equiv -\frac{1}{Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\frac{\delta}{\delta\eta_\beta(y)}Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])\Big|_{\bar{\eta}=\eta=0}. \end{aligned} \tag{4.3.2}$$

From (4.2.22), (4.2.23) and (4.2.24), satisfied by $Z_{\mathrm{GC}}(\beta;[J,\bar{\eta},\eta])$, in a similar manner as in Sect. 2.4, we obtain the Schwinger–Dyson equation satisfied by $D^J(x,y)$ and $G^J_{\alpha,\beta}(x,y)$ as

$$\begin{aligned} &(\mathrm{i}\gamma^\nu\partial_\nu - m + g\gamma\langle\hat{\phi}(x)\rangle^J)_{\alpha\varepsilon}G^J_{\varepsilon\beta}(x,y) - \int \mathrm{d}^4z\,\boldsymbol{\Sigma}^*_{\alpha\varepsilon}(x,z)G^J_{\varepsilon\beta}(z,y) \\ &= \delta_{\alpha\beta}\delta^4(x-y), \end{aligned} \tag{4.3.3a}$$

$$\begin{aligned} &\left(\frac{\partial^2}{\partial x_\nu^2} - \kappa^2\right)\langle\hat{\phi}(x)\rangle^J \\ &= \tfrac{1}{2}g\gamma_{\beta\alpha}\{G^J_{\alpha\beta}(\tau,\boldsymbol{x};\tau-\varepsilon,\boldsymbol{x}) + G^J_{\alpha\beta}(\tau,\boldsymbol{x};\tau+\varepsilon,\boldsymbol{x})\}|_{\varepsilon\to0^+}, \end{aligned} \tag{4.3.3b}$$

$$\left(\frac{\partial^2}{\partial x_\nu^2} - \kappa^2\right)D^J(x,y) - \int \mathrm{d}^4z\,\boldsymbol{\Pi}^*(x,z)D^J(z,y) = \delta^4(x-y)\ , \tag{4.3.3c}$$

$$\boldsymbol{\Sigma}^*_{\alpha\beta}(x,y) = g^2\int \mathrm{d}^4u\mathrm{d}^4v\,\gamma_{\alpha\delta}G^J_{\delta\nu}(x,u)\boldsymbol{\Gamma}_{\nu\beta}(u,y;v)D^J(v,x), \tag{4.3.3d}$$

$$\boldsymbol{\Pi}^*(x,y) = g^2\int \mathrm{d}^4u\mathrm{d}^4v\,\gamma_{\alpha\beta}G^J_{\beta\delta}(x,u)\boldsymbol{\Gamma}_{\delta\nu}(u,v;y)G^J_{\nu\alpha}(v,x), \tag{4.3.3e}$$

$$\boldsymbol{\Gamma}_{\alpha\beta}(x,y;z)=\gamma_{\alpha\beta}(z)\delta^4(x-y)\delta^4(x-z)+\frac{1}{g}\frac{\delta\boldsymbol{\Sigma}^*_{\alpha\beta}(x,y)}{\delta\langle\hat{\phi}(z)\rangle^J}. \tag{4.3.3f}$$

We here employ the abbreviations

$$x\equiv(\tau_x,\boldsymbol{x}),\qquad \int \mathrm{d}^4x\equiv\int_0^\beta \mathrm{d}\tau_x\int \mathrm{d}^3\boldsymbol{x}. \tag{4.3.4}$$

We note that $G^J_{\alpha\beta}(x,y)$ and $D^J(x,y)$ are determined by (4.3.3a) through (4.3.3f) only for

$$\tau_x-\tau_y\in[-\beta,\beta],$$

and we assume that they are defined by a periodic boundary condition with period $2\beta$ for

$$\tau_x-\tau_y\notin[-\beta,\beta].$$

Next, we set

$$J\equiv 0,$$

and hence we have

$$\langle\hat{\phi}(x)\rangle^{J\equiv 0}\equiv 0,$$

restoring the translational invariance of the system. We Fourier transform $G_{\alpha\beta}(x)$ and $D(x)$,

$$G_{\alpha\beta}(x)=\frac{1}{\beta}\sum_{p_4}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}G_{\alpha\beta}(p_4,\boldsymbol{p})\exp\left[\mathrm{i}(\boldsymbol{p}\boldsymbol{x}-p_4\tau_x)\right], \tag{4.3.5a}$$

$$D(x)=\frac{1}{\beta}\sum_{p_4}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}D(p_4,\boldsymbol{p})\exp\left[\mathrm{i}(\boldsymbol{p}\boldsymbol{x}-p_4\tau_x)\right], \tag{4.3.5b}$$

$$\begin{aligned}\boldsymbol{\Gamma}_{\alpha,\beta}(x,y;z)&=\boldsymbol{\Gamma}_{\alpha,\beta}(x-y,x-z)\\&=\frac{1}{\beta^2}\sum_{p_4,k_4}\int\frac{\mathrm{d}^3\boldsymbol{p}\mathrm{d}^3\boldsymbol{k}}{(2\pi)^6}\boldsymbol{\Gamma}_{\alpha,\beta}(p,k)\\&\quad\times\exp\left[\mathrm{i}\{\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-p_4(\tau_x-\tau_y)\}\right.\\&\quad\left.-\mathrm{i}\{\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{z})-k_4(\tau_x-\tau_z)\}\right],\end{aligned} \tag{4.3.5c}$$

$$p_4=\begin{cases}(2n+1)\pi/\beta, & \text{fermion},\quad n=\text{ integer},\\ 2n\pi/\beta, & \text{boson},\quad n=\text{ integer}.\end{cases} \tag{4.3.5d}$$

We have the Schwinger–Dyson equation in momentum space as

$$\{-\boldsymbol{\gamma}\boldsymbol{p} + \gamma^4(p_4 - \mathrm{i}\mu) - (m + \boldsymbol{\Sigma}^*(p))\}_{\alpha\varepsilon} G_{\varepsilon\beta}(p) = \delta_{\alpha\beta} \sum_n \delta\left(p_4 - \frac{(2n+1)\pi}{\beta}\right), \tag{4.3.6a}$$

$$\{-k_\nu^2 - (\kappa^2 + \boldsymbol{\Pi}^*(k))\} D(k) = \sum_n \delta\left(k_4 - \frac{2n\pi}{\beta}\right), \tag{4.3.6b}$$

$$\boldsymbol{\Sigma}^*_{\alpha,\beta}(p) = g^2 \frac{1}{\beta} \sum_{k_4} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \gamma_{\alpha\delta} G_{\delta\varepsilon}(p+k) \boldsymbol{\Gamma}_{\varepsilon\beta}(p+k, k) D(k), \tag{4.3.6c}$$

$$\boldsymbol{\Pi}^*(k) = g^2 \frac{1}{\beta} \sum_{p_4} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \gamma_{\mu\nu} G_{\nu\lambda}(p+k) \boldsymbol{\Gamma}_{\lambda\rho}(p+k, k) G_{\rho\mu}(p), \tag{4.3.6d}$$

$$\boldsymbol{\Gamma}_{\alpha\beta}(p,k) = \sum_{n,,m} \gamma_{\alpha\beta} \delta\left(p_4 - \frac{(2n+1)\pi}{\beta}\right) \delta\left(k_4 - \frac{(2m+1)\pi}{\beta}\right) + \boldsymbol{\Lambda}_{\alpha\beta}(p,k)\ , \tag{4.3.6e}$$

where $\boldsymbol{\Lambda}_{\alpha\beta}(p,k)$ represents the sum of the vertex diagram except for the first term.

### 4.3.2 Nonrelativistic Limit

We consider the nonrelativistic limit of the fermion "free" temperature Green's function,

$$G^{\ 0}_{\alpha,\beta}(p) = \sum_n \delta\left(p_4 - \frac{(2n+1)\pi}{\beta}\right) \left(\frac{1}{-\boldsymbol{\gamma}\boldsymbol{p} + \gamma^4(p_4 - \mathrm{i}\mu) - m}\right)_{\alpha,\beta} . \tag{4.3.7}$$

Multiplying by $\{-\boldsymbol{\gamma}\boldsymbol{p} + \gamma^4(p_4 - \mathrm{i}\mu) + m\}$ on the numerator and the denominator of the right-hand side of (4.3.7), we have

$$G^{\ 0}_{\alpha,\beta}(p) = \sum_n \{-\boldsymbol{\gamma}\boldsymbol{p} + \gamma^4(p_4 - \mathrm{i}\mu) + m\}_{\alpha,\beta} \times \frac{\delta(p_4 - (2n+1)\pi/\beta)}{-\{\boldsymbol{p}^2 + (p_4 - \mathrm{i}\mu)^2 + m^2\}} . \tag{4.3.8}$$

Upon taking the nonrelativistic limit, we have

$$\mu = m + \mu_1, \qquad \boldsymbol{\gamma} = 0, \qquad \gamma^4_{\alpha\beta} = \mathrm{i}\delta_{\alpha\beta}, \tag{4.3.9a}$$

$$\frac{p_4}{m} = \frac{(2n+1)\pi}{\beta m} \ll 1, \qquad \text{low temperature}, \tag{4.3.9b}$$

$$\frac{\boldsymbol{p}^2}{m^2} \ll 1, \qquad \frac{\mu_1}{m} \ll 1. \tag{4.3.9c}$$

We then have

$$\begin{aligned}
\text{numerator} &= 0 + \mathrm{i}\delta_{\alpha\beta}(p_4 - \mathrm{i}\mu) + m\delta_{\alpha\beta} \\
&= m\delta_{\alpha\beta}\left\{\mathrm{i}\frac{p_4}{m} + \left(1 + \frac{\mu_1}{m}\right) + 1\right\} \\
&\approx 2m\delta_{\alpha\beta}, \quad \text{(NR limit.)}
\end{aligned} \tag{4.3.10}$$

while

$$\begin{aligned}
\text{denominator} &= -(\boldsymbol{p}^2 + m^2) + (m + \mu_1 + ip_4)^2 \\
&= -\boldsymbol{p}^2 + 2m(\mu_1 + \mathrm{i}p_4) + \mu_1^2 + 2\mathrm{i}\mu_1 p_4 - p_4^2 \\
&\approx 2m\left(\mathrm{i}p_4 + \mu_1 - \frac{\boldsymbol{p}^2}{2m}\right). \quad \text{(NR limit.)}
\end{aligned} \tag{4.3.11}$$

Hence, we obtain

$$\begin{aligned}
G^{\mathrm{NR}}_{0\,\alpha,\beta}(p) = \; & \delta_{\alpha\beta}\sum_n \delta(p_4 - (2n+1)\pi/\beta) \\
& \times \frac{1}{\mathrm{i}p_4 + \mu_1 - \boldsymbol{p}^2/2m},
\end{aligned} \tag{4.3.12}$$

which agrees with the "free" temperature Green's function, (4.1.17), aside from the spinor indices, $\delta_{\alpha\beta}$, and the restriction of $p_4$. This validates the analytic continuation employed in Sect. 4.1.

## 4.4 Methods of the Auxiliary Field

We frequently encounter four-fermion interactions in the theory of condensed matter physics. In some theoretical calculations, we would like to change the form of the coupling in order to have a new perspective on the theory. Also, instead of dealing with the four-fermion interaction, we would like to change the form of the interaction to a bilinear form in the fermion fields and thus reduce the problem to the calculation of the fermion determinant. We can accomplish this goal with the method of the auxiliary fields, and we rewrite the four-fermion interaction in terms of the tri-linear Yukawa coupling of the physical fields $\{\psi_\sigma(x), \bar{\psi}_\sigma(x)\}$ and the auxiliary field.

In the method of the auxiliary field in the Lagrangian formalism (Sect. 4.4.1), we introduce the auxiliary field $\phi(x)$ on an equal footing to the physical fields $\{\psi_\sigma(x), \bar{\psi}_\sigma(x)\}$. The physical significance of the auxiliary field $\phi(x)$ lies in the fact that the Green's function of the auxiliary field $\phi(x)$ is a physical quantity related to the linear response function of the system.

In the method of the auxiliary field in the Hamiltonian formalism, we have the Gaussian method which can be applied to the quartic coupling of bosons and fermions (Sect. 4.4.2). The physical significance of the auxiliary field in the Hamiltonian formalism is not so clear as compared to that of the Lagrangian formalism.

### 4.4.1 Method of the Auxiliary Field in the Lagrangian Formalism

We consider the grand canonical ensemble described by the Euclidean Lagrangian density

$$\mathcal{L}'_{\mathrm{E}}(\psi(x), \partial_\mu\psi(x), \psi^*(x), \partial_\mu\psi^*(x)).$$

We let the Euclidean action functional $I[\psi, \psi^*]$ of this system be given by

$$\begin{aligned} I[\psi, \psi^*] &= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\mathcal{L}'_{\mathrm{E}}(\psi(x), \partial_\mu\psi(x), \psi^*(x), \partial_\mu\psi^*(x)) \\ &= I_0[\psi, \psi^*] + I_2[\psi, \psi^*] \,, \end{aligned} \tag{4.4.1}$$

where

$$I_0[\psi, \psi^*] = \text{ Euclidean action functional for a noninteracting system,}$$

and

$$I_2[\psi, \psi^*] = -\frac{1}{2}\int \mathrm{d}\tau_x \mathrm{d}^3\boldsymbol{x} \int \mathrm{d}\tau_y \mathrm{d}^3\boldsymbol{y}\lambda(x)K(x, y)\lambda(y), \tag{4.4.2}$$

where $\lambda(x)$ is given by

$$\lambda(x) = \sum_{n=2} C_n\psi_1^*(x)\cdots\psi_n(x). \tag{4.4.3}$$

By the introduction of the auxiliary field $\phi(x)$, we replace the quadratic term in $\lambda(x)$ which represents the self-interaction of $\psi(x)$ in $I_2[\psi, \psi^*]$ with the effective interaction $I_1'[\psi, \psi^*, \phi]$ which is bilinear in $\lambda(x)$ and $\phi(x)$. We can accomplish this by incompleting the square in the Gaussian functional integration,

$$\exp\left[-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y a(x)K(x,y)a(y)\right]$$
$$= \frac{\int \mathcal{D}[\phi] \exp\left[\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\phi(x)K^{-1}(x,y)\phi(y) - \int \mathrm{d}^4x a(x)\phi(x)\right]}{\int \mathcal{D}[\phi] \exp\left[\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\phi(x)K^{-1}(x,y)\phi(y)\right]} \,. \tag{4.4.4}$$

From (4.4.2) and (4.4.4), we obtain

$$\exp[I_2[\psi,\psi^*]]$$
$$= \frac{\int \mathcal{D}[\phi] \exp\left[\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\phi(x)K^{-1}(x,y)\phi(y) - \int \mathrm{d}^4x\lambda(x)\phi(x)\right]}{\int \mathcal{D}[\phi] \exp\left[\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\phi(x)K^{-1}(x,y)\phi(y)\right]} \,. \tag{4.4.5}$$

From (4.4.5), we obtain the effective Euclidean action functional

$$I_{\mathrm{eff}}[\psi,\psi^*,\phi]$$

of the $\psi(x)$-$\phi(x)$ system as

$$I_{\mathrm{eff}}[\psi,\psi^*,\phi]$$
$$= I_0[\psi,\psi^*] + \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\phi(x)K^{-1}(x,y)\phi(y) - \int \mathrm{d}^4x\lambda(x)\phi(x)$$
$$= I_0'[\psi,\psi^*,\phi] + I_1'[\psi,\psi^*,\phi], \tag{4.4.6a}$$

where

$$I_0'[\psi,\psi^*,\phi] = I_0[\psi,\psi^*] + \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\phi(x)K^{-1}(x,y)\phi(y), \tag{4.4.6b}$$

and

$$I_1'[\psi,\psi^*,\phi] = -\int \mathrm{d}^4x\lambda(x)\phi(x). \tag{4.4.6c}$$

We have the grand canonical ensemble average of the $\mathrm{T}_\tau$-ordered product of the Heisenberg picture operators, $\hat{B}(1)$, $\cdots$,$\hat{C}(n)$, in the Matsubara representation as

$$\langle \mathrm{T}_\tau\{\hat{B}(1)\cdots\hat{C}(n)\}\rangle$$
$$= \frac{\int \mathcal{D}[\psi]\mathcal{D}[\psi^*]\mathcal{D}[\phi]B(1)\cdots C(n)\exp[I_{\mathrm{eff}}[\psi,\psi^*,\phi]]}{\int \mathcal{D}[\psi]\mathcal{D}[\psi^*]\mathcal{D}[\phi]\exp[I_{\mathrm{eff}}[\psi,\psi^*,\phi]]} \,. \tag{4.4.7}$$

In particular, when we choose the auxiliary field $\phi(x)$ itself as $\hat{B}(1)$, $\cdots$ , we have the "full" and "free" temperature Green's functions of the auxiliary field $\phi(x)$ as

$$D^{\text{auxi.}}(x,y) = \frac{\int \mathcal{D}[\psi]\mathcal{D}[\psi^*]\mathcal{D}[\phi]\phi(x)\phi(y)\exp[I_{\text{eff}}[\psi,\psi^*,\phi]]}{\int \mathcal{D}[\psi]\mathcal{D}[\psi^*]\mathcal{D}[\phi]\exp[I_{\text{eff}}[\psi,\psi^*,\phi]]}\,, \tag{4.4.8}$$

$$D_0^{\text{auxi.}}(x,y) = \left\{ \begin{matrix} I_{\text{eff}}[\psi,\psi^*,\phi] \rightarrow I_0'[\psi,\psi^*,\phi] \\ \text{in (4.4.8)} \end{matrix} \right\} = K(x,y). \tag{4.4.9}$$

The physical significance of the Green's function of the auxiliary field $\phi(x)$ lies in the fact that it provides a linear response function of the $\psi(x)$ system. In the discussion so far, we have assumed the reality of $\lambda(x)$. When $\lambda(x)$ is complex, we introduce the complex auxiliary fields, $\phi(x)$ and $\phi^*(x)$, with the appropriate modifications in (4.4.4) through (4.4.9).

A noteworthy point of the method of the auxiliary field in the Lagrangian formalism lies in the fact that the auxiliary field $\phi(x)$ is treated on the equal footing with the physical fields $\psi(x)$ and $\psi^*(x)$, and the linked cluster expansion of the Green's function treats $\phi(x)$ and $\psi(x)$ on an equal footing. In this regard, the auxiliary field $\phi(x)$ in the Lagrangian formalism is distinct from the Gaussian auxiliary field $z(\tau)$ in the Stratonovich–Hubbard transformation in the Hamiltonian formalism. We usually interpret the $z(\tau)$ as the "time"-dependent external field and we average the physical quantity with respect to the functional Gaussian measure

$$\mathcal{D}[z]\mathcal{D}[z^*]\exp\left[-\frac{1}{\beta}\int_0^\beta \mathrm{d}\tau |z(\tau)|^2\right].$$

The dynamical content of $z(\tau)$ is not quite clear.

### 4.4.2 Stratonovich–Hubbard Transformation: Gaussian Method

Stratonovich and Hubbard considered the replacement of the partition function of the self-interacting bosonic (fermionic) many-body system by the partition function of the bosonic (fermionic) many-body system interacting with the "time"-dependent external field $z(\tau)$, with respect to which the Gaussian average is performed. The Gaussian average is taken over all possible external fields and is expressed in terms of the functional integral. The Hamiltonian formalism is employed for the original bosonic (fermionic) many-body system. The auxiliary bosonic field $z(\tau)$, with respect to which the Gaussian average is taken, is not treated on an equal footing with the bosonic (fermionic) field of the many-body system.

The starting point of this method is the following Gaussian integral formula,

$$\begin{aligned}\exp[-\hat{A}^2] &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{d}y \exp[-y^2 - 2\mathrm{i}y\hat{A}] \\ &\equiv \langle \exp[-2\mathrm{i}y\hat{A}]\rangle_{\text{Gauss. Ave.}}.\end{aligned} \tag{4.4.10}$$

We accomplished the linearization of the operator $\hat{A}^2$ with the use of a one-dimensional Gaussian average. As for the two commuting operators, $\hat{A}$ and $\hat{B}$, since the product $\hat{A}\cdot\hat{B}$ can be expressed as

$$-\hat{A}\cdot\hat{B} = -\left(\frac{\hat{A}+\hat{B}}{2}\right)^2 + \left(\frac{\hat{A}-\hat{B}}{2}\right)^2 , \quad [\hat{A},\hat{B}] = 0, \tag{4.4.11}$$

we use (4.4.10) for each squared term in (4.4.11) and obtain

$$\exp[-\hat{A}\cdot\hat{B}]$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}x\mathrm{d}y \exp[-(x^2+y^2) - x(\hat{A}-\hat{B}) - \mathrm{i}y(\hat{A}+\hat{B})] \tag{4.4.12}$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}z\mathrm{d}z^* \exp[-|z|^2 - z\hat{A} + z^*\hat{B}] \tag{4.4.13a}$$

$$\equiv \langle \exp[-z\hat{A} + z^*\hat{B}]\rangle_{\text{Gauss. Ave.}} . \tag{4.4.13b}$$

We accomplished the linearization of the product $\hat{A}\cdot\hat{B}$ of the two commuting operators $\hat{A}$ and $\hat{B}$ with the use of the two dimensional Gaussian average.

With these preparations, we consider a system described by the following grand canonical Hamiltonian

$$\hat{H}' = (\hat{H}_0 - \mu\hat{N}) + \hat{A}\cdot\hat{B}, \tag{4.4.14a}$$

$$[\hat{A},\hat{B}] = 0 , \quad [\hat{H}_0 - \mu\hat{N}, \hat{A}] \neq 0 , \quad [\hat{H}_0 - \mu\hat{N}, \hat{B}] \neq 0 . \tag{4.4.14b}$$

According to the Fradkin construction in Sect. 4.2.1, (4.2.6) through (4.2.10), we have the density matrix for the grand canonical ensemble as

$$\begin{aligned} \hat{\rho}_{\mathrm{GC}}(\beta) &= \exp[-\beta(\hat{H}_0 - \mu\hat{N} + \hat{A}\cdot\hat{B})] \\ &= \exp[-\beta(\hat{H}_0 - \mu\hat{N})]\mathrm{T}_\tau\left\{\exp\left[-\int_0^\beta \mathrm{d}\tau \hat{A}^{(\mathrm{I})}(\tau)\hat{B}^{(\mathrm{I})}(\tau)\right]\right\} , \end{aligned} \tag{4.4.15}$$

where

$$\hat{A}^{(\mathrm{I})}(\tau) = \exp[\tau(\hat{H}_0 - \mu\hat{N})]\hat{A}\exp[-\tau(\hat{H}_0 - \mu\hat{N})] . \tag{4.4.16}$$

Since we can freely change the order of the operators under the $\mathrm{T}_\tau$-ordering symbol, we can apply (4.4.12), (4.4.13a) and (4.4.13b) to the { } part of (4.4.15).

$$
\begin{aligned}
\hat{E}(\tau) &\equiv \exp\left[-\int_0^\beta \mathrm{d}\tau \hat{A}(\tau)\hat{B}(\tau)\right] \\
&= \lim_{\substack{\Delta\tau=\frac{\beta}{N}\to 0\\ N\to+\infty}} \exp\left[-\sum_{j=1}^{N} \Delta\tau \hat{A}(\tau_j)\hat{B}(\tau_j)\right] \quad (\tau_j = j\Delta\tau) \qquad (4.4.17\mathrm{a})
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\substack{\Delta\tau\to 0\\ N\to+\infty}} \left(\frac{1}{\pi}\right)^N \int \prod_{j=1}^{N} \left(\frac{\Delta\tau}{\beta}\mathrm{d}z_j\right)\left(\frac{\Delta\tau}{\beta}\mathrm{d}z_j^*\right) \\
&\quad \times \exp\left[-\frac{1}{\beta}\sum_{j=1}^{N} \Delta\tau\{|z_j|^2 + \hat{A}(\tau_j)z_j - \hat{B}(\tau_j)z_j^*\}\right] \\
&= \lim_{\substack{\Delta\tau\to 0\\ N\to+\infty}} \int \prod_{j=1}^{N} \left(\frac{\Delta\tau}{\sqrt{\pi}\beta}\mathrm{d}z_j\right)\left(\frac{\Delta\tau}{\sqrt{\pi}\beta}\mathrm{d}z_j^*\right) \\
&\quad \times \exp\left[-\frac{1}{\beta}\sum_{j=1}^{N} \Delta\tau\{|z_j|^2 + \hat{A}(\tau_j)z_j - \hat{B}(\tau_j)z_j^*\}\right] \\
&= \int \mathcal{D}[z]\mathcal{D}[z^*] \\
&\quad \times \exp\left[-\frac{1}{\beta}\int_0^\beta \mathrm{d}\tau\{|z(\tau)|^2 + \hat{A}(\tau)z(\tau) - \hat{B}(\tau)z^*(\tau)\}\right], \quad (4.4.17\mathrm{b})
\end{aligned}
$$

where the functional integral measure is given by

$$
\begin{aligned}
&\mathcal{D}[z]\mathcal{D}[z^*]\exp\left[-\frac{1}{\beta}\int_0^\beta d\tau|z(\tau)|^2\right] \\
&= \lim_{\substack{\Delta\tau=\frac{\beta}{N}\to 0\\ N\to+\infty}} \prod_{j=1}^{N} \left(\frac{\Delta\tau}{\sqrt{\pi}\beta}\mathrm{d}z_j\right)\left(\frac{\Delta\tau}{\sqrt{\pi}\beta}\mathrm{d}z_j^*\right)\exp\left[-\frac{1}{\beta}\sum_{j=1}^{N}\Delta\tau|z_j|^2\right]. \quad (4.4.18)
\end{aligned}
$$

Returning to (4.4.15), we exchange the order of the $\mathrm{T}_\tau$-ordering symbol and the functional integral

$$
\int \mathcal{D}[z]\mathcal{D}[z^*],
$$

and obtain

$$
\begin{aligned}
&\hat{\rho}_{\mathrm{GC}}(\beta) \\
&= \exp[-\beta(\hat{H}_0 - \mu\hat{N})]\mathrm{T}_\tau\hat{E}(\tau)
\end{aligned}
$$

$$= \int \mathcal{D}[z]\mathcal{D}[z^*] \exp\left[-\frac{1}{\beta}\int_0^\beta \mathrm{d}\tau |z(\tau)|^2\right] \exp[-\beta(\hat{H}_0 - \mu\hat{N})]$$
$$\times \mathrm{T}_\tau \left\{\exp\left[-\frac{1}{\beta}\int_0^\beta \mathrm{d}\tau(\hat{A}^{(\mathrm{I})}(\tau)z(\tau) - \hat{B}^{(\mathrm{I})}(\tau)z^*(\tau))\right]\right\} \tag{4.4.19a}$$

$$= \Bigg\langle \exp[-\beta(\hat{H}_0 - \mu\hat{N})] \times$$
$$\times \mathrm{T}_\tau \left\{\exp\left[-\frac{1}{\beta}\int_0^\beta \mathrm{d}\tau(\hat{A}^{(\mathrm{I})}(\tau)z(\tau) - \hat{B}^{(\mathrm{I})}(\tau)z^*(\tau))\right]\right\} \Bigg\rangle_{\text{Funct.Gauss.Ave.}} . \tag{4.4.19b}$$

Here, we note the following correspondence,

$$z(\tau) = x(\tau) + \mathrm{i}y(\tau) \leftrightarrow z = x + \mathrm{i}y \text{ of (4.4.13a and b)},$$

and

$$\text{functional Gaussian average} \leftrightarrow \text{Gaussian average in (4.4.13a and b)}.$$

Equations (4.4.19a) and (4.4.19b) indicate that the linearization of the product $\hat{A}\cdot\hat{B}$ of operators satisfying (4.4.14b) is accomplished with the "two-dimensional" functional Gaussian average. We can combine the two exponents of (4.4.19b) into one by

$$\hat{\rho}_{\mathrm{GC}}(\beta) = \Bigg\langle \mathrm{T}_\tau \exp\Bigg[-\int_0^\beta \mathrm{d}\tau\Bigg\{(\hat{H}_0 - \mu\hat{N})_\tau$$
$$+\frac{1}{\beta}(\hat{A}^{(\mathrm{I})}(\tau)z(\tau) - \hat{B}^{(\mathrm{I})}(\tau)z^*(\tau))\Bigg\}\Bigg]\Bigg\rangle_{\mathrm{F.G.A.}} , \tag{4.4.20}$$

where F.G.A. indicates the functional Gaussian average and $(\hat{H}_0 - \mu\hat{N})_\tau$ represents

$$\begin{aligned}(\hat{H}_0 - \mu\hat{N})_\tau &\equiv \exp[\tau(\hat{H}_0 - \mu\hat{N})](\hat{H}_0 - \mu\hat{N})\exp[-\tau(\hat{H}_0 - \mu\hat{N})] \\ &= \hat{H}_0 - \mu\hat{N} \, .\end{aligned} \tag{4.4.21}$$

The formula (4.4.20) is called the *Stratonovich–Hubbard identity.*

If $\hat{A}$ and $\hat{B}$ commute with $\hat{H}_0 - \mu\hat{N}$,

$$[\hat{A}, \hat{H}_0 - \mu\hat{N}] = [\hat{B}, \hat{H}_0 - \mu\hat{N}] = 0, \tag{4.4.22}$$

we have

$$\hat{A}^{(\mathrm{I})}(\tau) = \hat{A}, \quad \hat{B}^{(\mathrm{I})}(\tau) = \hat{B}, \tag{4.4.23}$$

so that the $T_\tau$-ordering becomes unnecessary. Then, $\hat{A}$ and $\hat{B}$ interact with

$$z_0 = \frac{1}{\beta}\int_0^\beta \mathrm{d}\tau z(\tau) \ , \quad z_0^* = \frac{1}{\beta}\int_0^\beta \mathrm{d}\tau z^*(\tau), \tag{4.4.24}$$

respectively, and the functional Gaussian average is reduced to the ordinary Gaussian average.

When $\hat{A}$ and $\hat{B}$ are given by the bilinear form of the fermion operators, $\hat{\psi}_\sigma$ and $(\hat{\psi}^\dagger\gamma^0)_\sigma$, for example,

$$\hat{A} = \hat{n}_\sigma = (\hat{\psi}^\dagger\gamma^0)_\sigma\hat{\psi}_\sigma \ , \quad \hat{B} = \hat{n}_{\sigma'} = (\hat{\psi}^\dagger\gamma^0)_{\sigma'}\hat{\psi}_{\sigma'}, \tag{4.4.25}$$

the four-fermion interaction gets linearized to the tri-linear Yukawa coupling of $\hat{\psi}_\sigma$, $(\hat{\psi}^\dagger\gamma^0)_\sigma$ and the auxiliary boson field $z(\tau)$. Finally, the problem is reduced to the fermion determinant, even in the Hamiltonian formalism, which depends on the auxiliary boson field $z(\tau)$. Hence it is subject to the functional Gaussian average. For the fermion determinant in the Hamiltonian formalism, the reader is referred to Blankenbecler *et al.* (1981).

## 4.5 Bibliography

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# 5. Stochastic Quantization

In this chapter, we discuss the stochastic quantization of the classical theory. We attempt to formulate quantum mechanics and quantum field theory in the **Euclidean metric** as a stationary state of a stochastic process. When we compare the composition law of the transition probabilities of the stochastic process with that of the transition probability amplitudes of quantum mechanics and quantum field theory, we notice that the evolution parameter of the stochastic process is missing in quantum mechanics and quantum field theory. If we identify the real time variable $t$ as the evolution parameter of the stochastic process, we do not obtain the Schrödinger equation. Hence, we must introduce an evolution parameter of the stochastic process into quantum mechanics and quantum field theory in the stochastic quantization method.

In the theory of probability, there are four modes of convergence:

A. strong convergence (almost sure/everywhere convergence),
B. convergence in probability,
C. weak convergence (convergence in distribution),
D. convergence in the mean square,

with the implications A⇒B⇒C and D⇒B. In stochastic quantization, we should clarify in which mode we have a convergence. In the theory of stochastic process, many distinct stochastic processes converge to the same stationary state, say, a Gaussian process.

In Sect. 5.1, we begin with a review of the theory of probability and stochastic processes, and write down the evolution equation of quantum mechanics and quantum field theory as a stochastic process. We first discuss random variables and the notion of convergence of random variables (Sect. 5.1.1). Secondly, we discuss stochastic processes, Kolmogoroff's consistency condition and Kolmogoroff's existence theorem (Sect. 5.1.2). Lastly, we compare the composition law of the transition probability of a stochastic process with that of the transition probability amplitude of quantum mechanics and quantum field theory. We introduce a finite evolution parameter of a stochastic process into quantum mechanics and quantum field theory, and write down the evolution equation of quantum mechanics and quantum field theory in terms of the finite evolution parameter, together with the requisite stationary conditions (Sect. 5.1.3).

M. Masujima: Path Integral Quantization and Stochastic Quantization,
STMP 165, 209–241 (2000)

In Sect. 5.2, with the use of the Fokker–Planck equation together with the evolution equation, we formulate quantum mechanics and quantum field theory in the Euclidean metric as a stationary state of a stochastic process (Sect. 5.2.1). Secondly, we discuss the stochastic quantization of Abelian and non-Abelian gauge fields. In the quantization of the non-Abelian gauge field in the standard approach, we introduce the gauge fixing term and the requisite Faddeev–Popov ghost term into the action functional as discussed in Chap. 3. In the stochastic quantization of non-Abelian gauge fields, we do not fix the gauge at the outset. The derivative term of the gauge field with respect to the evolution parameter is present in addition to the four-dimensional transverse projection operator of the gauge field in the evolution equation. The evolution equation is invertible for the gauge field without the gauge-fixing term. Instead of inverting the evolution equation, we split the evolution equation into a transverse mode and a longitudinal mode of the gauge field. We calculate the correlation function of the gauge field. From this, we obtain the gauge field "free" Green's function. We discuss the initial distribution of the longitudinal mode of the gauge field. The width of the initial symmetric distribution of the longitudinal mode provides the gauge parameter of the covariant gauge "free" Green's function. The evolution equation for the transverse mode provides the usual gluon contribution, whereas the evolution equation for the longitudinal mode provides the Faddeev–Popov ghost effect, thus restoring unitarity (Sect. 5.2.2). We discuss the covariant nonholonomic gauge-fixing condition, which cannot be written as an addition to the action functional of non-Abelian gauge fields. We discuss the Fokker–Planck equation for non-Abelian gauge field theory with the covariant nonholonomic gauge-fixing condition. We demonstrate that non-Abelian gauge field theory can be seen as a stationary state of a stochastic process (Sect. 5.2.3).

The stochastic quantization method is the equation of motion approach to $c$-number quantization, and in principle requires neither the Hamiltonian nor the Lagrangian. It has a wider applicability than canonical quantization and path integral quantization. It may also provide quantization of a nonholonomic constraint system. This is the case for the stochastic quantization of non-Abelian gauge field theory with the nonholonomic gauge-fixing condition.

## 5.1 Review of the Theory of Probability and Stochastic Processes

In this section, we review the theory of probability and stochastic processes, and derive the evolution equation of quantum mechanics and quantum field theory. First, we discuss random variables and the notion of convergence of random variables (Sect. 5.1.1). We begin with the probability space, the conditional probability, the independence of probabilistic events, Bayes' theorem, random variables, the law of the probability distribution, the cumulative distribution function, the expectation value, the moment, the moment generating function and the Tchebychev inequality. Then we discuss the relationships among strong convergence (almost sure convergence), convergence in probability, weak convergence (convergence in distribution) and convergence in the mean square. Secondly, we discuss the theory of stochastic processes, Kolmogoroff's consistency condition and Kolmogoroff's existence theorem (Sect. 5.1.2). Lastly, we compare the composition law of the transition probability of stochastic processes with that of the transition probability amplitude of quantum mechanics and quantum field theory. We introduce a finite evolution parameter of a stochastic process into Euclidean quantum mechanics and Euclidean quantum field theory, and write down the evolution equation of quantum mechanics and quantum field theory in terms of the finite evolution parameter, together with the requisite stationary conditions (Sect. 5.1.3).

### 5.1.1 Random Variables and the Notion of Convergence

We usually think of probability as the long-time average of the frequency of success of some event (experiment). Mathematically, we shall formulate the theory of probability based on measure theory.

As the sample space $\Omega$, we take the set of possible outcomes $\omega$ of the experiment. The **$\sigma$-algebra** $\mathcal{S}$ of a subset of $\Omega$ is the algebra with the following properties,

$$\text{(a)}\ \Omega \in \mathcal{S}, \tag{5.1.1a}$$

$$\text{(b)}\ \forall E \in \mathcal{S}, \quad \exists E^C \equiv \Omega \setminus E \in \mathcal{S}, \tag{5.1.1b}$$

$$\text{(c)}\ \forall E_n \in \mathcal{S},\ n = 1, 2, \ldots, \quad \cup_n E_n \in \mathcal{S}. \tag{5.1.1c}$$

The probability measure $P$, defined with respect to the event $E(\subset \Omega)$ which is an element of $\mathcal{S}$, is the mapping from $\Omega$ onto $[0, 1]$ with the following properties,

$$\text{(1)}\ P(\Omega) = 1, \tag{5.1.2a}$$

$$\text{(2)}\ \forall E \in \mathcal{S}, \quad 0 \leq P(E) \leq 1, \tag{5.1.2b}$$

$$\text{(3)}\ E_n \in S,\ n = 1, 2, \ldots, \infty, \quad E_i \cap E_j = \varnothing \quad (i \neq j),$$

$$P(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n). \tag{5.1.2c}$$

We call $(\Omega, \mathcal{S}, P)$ the probability space. From (5.1.1a) through (5.1.2c), we have

$$P(E) \leq P(F) \quad \text{if} \quad E \subset F \in \mathcal{S}, \tag{5.1.3a}$$

$$P(E) + P(E^C) = 1 \quad \text{if} \quad E \in \mathcal{S}, \tag{5.1.3b}$$

$$P(\varnothing) = 0, \qquad \text{for the empty set } \varnothing \in \mathcal{S}, \tag{5.1.3c}$$

$$P(\cup_n E_n) \leq \sum_n P(E_n) \quad \text{if} \quad E_n \in \mathcal{S}, n = 1, 2, \ldots, \tag{5.1.3d}$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \quad \text{for} \quad E_1, E_2 \in \mathcal{S}. \tag{5.1.3e}$$

The conditional probability of the event $E$ given the event $F$ is defined by

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \quad \text{if} \quad P(F) \neq 0, \tag{5.1.4}$$

and hence we have

$$P(E \cap F) = P(F)P(E \mid F) = P(E)P(F \mid E). \tag{5.1.5}$$

We say that the events $E$ and $F$ are independent when

$$P(E \cap F) = P(E)P(F). \tag{5.1.6}$$

We say that the events $\{E_1, E_2, \ldots, E_n\}$ are mutually independent when

$$\begin{aligned} &P(\cap_{k=1} E_{i_k}) \\ &= \prod_{k=1} P(E_{i_k}) \text{ for every subset of distinct integers } (i_1, i_2, \ldots, i_n). \end{aligned} \tag{5.1.7}$$

If $\{A_i\}_{i=1}^n$ is a disjoint partition of $\Omega$,

$$\cup_{i=1}^n A_i = \Omega, \quad A_i \cap A_j = \varnothing \quad (i \neq j),$$

we have

$$P(B) = \sum_{i=1}^n P(A_i)P(B \mid A_i). \tag{5.1.8}$$

Then, we have Bayes' theorem,

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^n P(A_j)P(B \mid A_j)} = \frac{P(A_i)P(B \mid A_i)}{P(B)}, \tag{5.1.9a}$$

$$\frac{P(A_i \mid B)}{P(A_j \mid B)} = \frac{P(A_i)}{P(A_j)} \cdot \frac{P(B \mid A_i)}{P(B \mid A_j)}. \tag{5.1.9b}$$

We call $X(\cdot)$ a random variable when the numerical value $X = X(\omega)$ corresponds to the outcome $\omega \in \Omega$ of the experiment. The law of the probability distribution, $\mathcal{L}(X)$, of the random variable $X$ is the restriction of the probability measure $P$ to the element $E$ of $\mathcal{S}$,

$$\mathcal{L}(X) = P \circ X^{-1}. \tag{5.1.10}$$

Ordinarily, we use the abbreviation

$$\{1 \leq X \leq 3\} \equiv \{\omega \in \Omega : 1 \leq X(\omega) \leq 3\}. \tag{5.1.11}$$

The law of the probability distribution $\mathcal{L}(X)$ is characterized by the cumulative distribution function. The cumulative distribution function of the random variable $X$ is defined by

$$F_X(x) \equiv P(X \leq x). \tag{5.1.12}$$

The cumulative distribution function has the following properties.

(1) $0 \leq F_X(x) \leq 1$. (5.1.13a)

(2) $F_X(x)$ is a monotonically nondecreasing function.
(3) $F_X(x)$ is right-continuous.

(4) $F_X(x) \to 0 \quad \text{as} \quad x \to -\infty$. (5.1.13b)

(5) $F_X(x) \to 1 \quad \text{as} \quad x \to +\infty$. (5.1.13c)

For a discrete random variable $X$, we have a sequence $\{x_i\}$ such that

$$\sum_i P\{X = x_i\} = 1. \tag{5.1.14}$$

We define the probability mass function $f_X(x)$ by

$$f_X(x) = P\{X = x\}, \tag{5.1.15}$$

with the following properties:

$$f_X(x) \geq 0, \quad \sum_{\{x_i\}} f_X(x_i) = 1, \quad F_X(x) = \sum_{\{x_i \leq x\}} f_X(x_i). \tag{5.1.16}$$

For a continuous random variable $X$, we have the probability density function $f_X(x)$ with the following properties:

$$f_X(x) \geq 0, \quad \int_{-\infty}^{+\infty} \mathrm{d}x f_X(x) = 1, \quad F_X(x) = \int_{-\infty}^{x} \mathrm{d}x f_X(x). \tag{5.1.17}$$

The cumulative distribution function of the continuous random variable is continuous and we have the following properties:

$$P_X(x) = 0 \quad \text{for} \quad \forall x \in \mathbb{R}, \tag{5.1.18a}$$

$$P\{a \leq X \leq b\} = \int_a^b f_X(x)\mathrm{d}x, \tag{5.1.18b}$$

$$P\{x \leq X \leq x + \mathrm{d}x\} = f_X(x)\mathrm{d}x, \tag{5.1.18c}$$

$$\frac{dF_X(x)}{\mathrm{d}x} = f_X(x). \tag{5.1.18d}$$

The expectation value $EX$ of the random variable $X(\cdot)$ is constructed such that the long-time average of $X(\cdot)$ will approximate the expectation value and is defined by

$$EX \equiv \int x\mathrm{d}F_X(x)$$

$$= \begin{cases} \sum_i x_i f_X(x_i) & \text{discrete r.v.,} \quad (5.1.19a) \\ \int_{-\infty}^{+\infty} x f_X(x)\mathrm{d}x & \text{continuous r.v.} \quad (5.1.19b) \end{cases}$$

Generally, for a function $h(X(\cdot))$ of a random variable $X(\cdot)$, we have

$$Eh(X) = \int h(x)\mathrm{d}F_X(x)$$

$$= \begin{cases} \sum_i h(x_i) f_X(x_i) & \text{discrete r.v.,} \quad (5.1.20a) \\ \int_{-\infty}^{+\infty} h(x) f_X(x)\mathrm{d}x & \text{continuous r.v.} \quad (5.1.20b) \end{cases}$$

We list the four basic properties of expectation values:

$$E[h_1(X) + h_2(X)] = Eh_1(X) + Eh_2(X). \tag{5.1.21a}$$

$$E[ch(X)] = cEh(X), \quad c \text{ constant.} \tag{5.1.21b}$$

$$E1 = 1. \tag{5.1.21c}$$

$$Eh(X) \geq 0 \quad \text{if} \quad h(X) \geq 0 \quad \text{with probability 1.} \tag{5.1.21d}$$

The $n^{\text{th}}$ moment $\mu_n'$ around the origin of the random variable $X$ is defined by

$$\mu_n' \equiv EX^n, \quad \mu_1' \equiv \mu_X \equiv EX = \text{mean of } X. \tag{5.1.22}$$

The $n^{\text{th}}$ moment $\mu_n$ around the mean $\mu_X$ of the random variable $X$ is defined by

$$\mu_n \equiv E(X - \mu_X)^n, \quad \mu_2 \equiv \sigma_X^2 \equiv E(X - \mu_X)^2 = \text{variance of } X. \tag{5.1.23}$$

The moment generating function $M_X(t)$ is defined by

$$M_X(t) \equiv E \exp\left[tX\right], \tag{5.1.24}$$

with the following properties:

$$\mu_n' = \left. \frac{\mathrm{d}^n}{\mathrm{d}x^n} M_X(t) \right|_{t=0}, \qquad \mu_n = \left. \frac{\mathrm{d}^n}{\mathrm{d}x^n} M_{X-\mu_X}(t) \right|_{t=0}. \tag{5.1.25}$$

The moment generating function has a property called "tilting". When the random variable $Y$ is given by

$$Y = a + bX, \tag{5.1.26}$$

the moment generating function of $Y$ is given by

$$M_Y(t) = \exp[at] M_X(bt). \tag{5.1.27}$$

We record here the *Tchebychev inequality* when the random variable $X$ has finite mean $\mu_X$ and finite variance $\sigma_X^2$,

$$P\left(\frac{|X - \mu_X|}{\sigma_X} \geq \lambda\right) \leq \frac{1}{\lambda^2}. \tag{5.1.28}$$

We have four definitions of the convergence of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$:

(A) strong convergence (almost sure convergence)
(B) convergence in probability
(C) weak convergence (convergence in distribution)
(D) convergence in mean square.

**Strong Convergence**: We abbreviate this as (a.s.), (ae.), or (w.p.1).

$$X_n \to X_0 \quad \text{as} \quad n \to \infty \quad \text{(a.s.), (ae.), or (w.p.1),}$$

if and only if

$$\exists A \in \mathcal{S} \text{ with } P(A) = 0 : \forall \omega \notin A,\ X_n(\omega) \to X_0(\omega) \quad \text{as} \quad n \to \infty. \tag{5.1.29}$$

**Convergence in Probability**: We abbreviate this as (in probability).

$$X_n \to X_0 \quad \text{as} \quad n \to \infty \quad \text{(in probability)},$$

if and only if

$$\forall \varepsilon > 0, \quad P(|X_n - X_0| > \varepsilon) < \varepsilon \quad \text{as} \quad n \to \infty. \tag{5.1.30}$$

**Weak Convergence**: We abbreviate this as (in distribution).

$$X_n \to X_0 \quad \text{as} \quad n \to \infty \quad \text{(in distribution)},$$

if and only if

$$\forall f(x) \in C_b(\mathbb{R}) \equiv \text{a set of all continuous bounded functions defined on } \mathbb{R},$$

$$\int f(x)\mathrm{d}F_n(x) \to \int f(x)\mathrm{d}F_0(x) \quad \text{as} \quad n \to \infty. \tag{5.1.31}$$

**Convergence in Mean Square**: We abbreviate this as (in m.s.).

$$X_n \to X_0 \quad \text{as} \quad n \to \infty \quad \text{(in m.s.)},$$

if and only if

$$E(X_n - X_0)^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{5.1.32}$$

The relationship among these modes of convergence is

$$\text{(A)} \Rightarrow \text{(B)} \Rightarrow \text{(C)}, \quad \text{and} \quad \text{(D)} \Rightarrow \text{(B)}. \tag{5.1.33}$$

### 5.1.2 Stochastic Processes

We call the collection of random variables, $\{X(t), t \in T\}$, defined on the probability space $(\Omega, \mathcal{S}, P)$ a stochastic process. We call $T$ the index set, whose element $t \in T$ is the evolution parameter of the stochastic process. We have two cases for $T$:

$$T = [0, \infty), \quad \text{or} \quad T = \{0, 1, 2, 3, \ldots\}. \tag{5.1.34}$$

We shall be concerned with the former $T$.

Since $X(t)$ is a random variable, we have

$$X(t) \equiv X(t, \omega), \quad \omega \in \Omega. \tag{5.1.35}$$

In the stochastic process $\{X(t,\omega)\}$, when $t$ is held fixed at $t = t_0$, we obtain a random variable $X(t_0,\omega)$, whereas when $\omega$ is held fixed at $\omega = \omega_0$, we obtain a function of $t \in T$, $X(t,\omega_0)$. We call the latter a sample function (or a sample path) for $\omega = \omega_0$.

We define the finite-dimensional distribution function $\mu_{t_1,\dots,t_k}(H)$, $H \in \mathbb{R}^k$, of the stochastic process $\{X(t), t \in T\}$ by

$$\mu_{t_1,\dots,t_k}(H) \equiv P(\{X(t_1),\dots,X(t_k)\} \in H), \quad H \in \mathbb{R}^k, \tag{5.1.36}$$

which necessarily satisfies *Kolmogoroff's consistency condition*:

$$(1)\ \mu_{t_1,\dots,t_k} = \mu_{t_{\pi 1},\dots,t_{\pi k}} \circ \varphi_\pi^{-1}, \tag{5.1.37}$$

$$(X(t_1),\dots,X(t_k)) \equiv \varphi_\pi(X(t_{\pi 1}),\dots,X(t_{\pi k})), \tag{5.1.38}$$

$$(2)\ \mu_{t_1,\dots,t_k}(H) = \mu_{t_1,\dots,t_k,t_{k+1}}(H \otimes \mathbb{R}). \tag{5.1.39}$$

The inverse of this consistency condition is *Kolmogoroff's existence theorem*:

When $\{\mu_{t_1,\dots,t_k}\}_{k\geq 1}$ satisfies the consistency conditions (1) and (2), there exists a stochastic process $\{X(t), t \in T\}$ on some probability space $(\Omega,\mathcal{S},P)$, whose finite-dimensional distribution function is $\{\mu_{t_1,\dots,t_k}\}_{k\geq 1}$.

(For a proof of the existence theorem, the reader is referred to Billingsley (1979).)

Among the many stochastic processes $\{X(t), t \in T\}$, the *Markov process* plays an important role. We call the stochastic process $\{X(t), t \in T\}$ a Markov process when the following condition holds,

$$\forall t_i \in T : t_{n+1} > t_n > \dots > t_2 > t_1,$$

$$P(X(t_{n+1}) = x_{t_{n+1}} \mid X(t_n) = x_{t_n},\dots,X(t_1) = x_{t_1})$$
$$= P(X(t_{n+1}) = x_{t_{n+1}} \mid X(t_n) = x_{t_n}) \qquad \text{(a.s.)}. \tag{5.1.40}$$

The future of the Markov process ($t = t_{n+1}$) depends only on the present ($t = t_n$), and not on the past ($t = t_j$, $j < n$). We call the conditional probability on the right-hand side of (5.1.40) the *transition probability* $W(X(t_{n+1}) = x_{t_{n+1}} \leftarrow X(t_n) = x_{t_n})$ of the Markov process:

$$W(X(t) = x_t \leftarrow X(s) = x_s) = P(X(t) = x_t \mid X(s) = x_s),\ t > s,\ \text{(a.s.)}. \tag{5.1.41}$$

We let the probability density function of the initial state of the Markov process be $p^{(0)}_{X(t_0)}(x_{t_0})$. The probability density function of the Markov process after the $n^{th}$ transition $p^{(n)}_{X(t_n)}(x_{t_n})$ is defined inductively by

$$
\begin{aligned}
&p^{(n)}_{X(t_n)}(x_{t_n}) \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x_{t_{n-1}} W(X(t_n) = x_{t_n} \leftarrow X(t_{n-1}) = x_{t_{n-1}}) p^{(n-1)}_{X(t_{n-1})}(x_{t_{n-1}}), \\
&n = 1, 2, \ldots .
\end{aligned} \tag{5.1.42}
$$

We let $n \to \infty$ at this stage. We obtain the equilibrium state (or the stationary state) of the Markov process, which is independent of the initial state. We have the probability density function $p^{(\mathrm{eq})}_{X(t)}(x_t)$ of the stationary state of the Markov process defined by

$$
p^{(\mathrm{eq})}_{X(t)}(x_t) \equiv \lim_{n\to\infty} p^{(n)}_{X(t)}(x_t), \quad -\infty < t < +\infty. \tag{5.1.43}
$$

In (5.1.42), we let $n \to \infty$ to obtain

$$
(\alpha)\ p^{(\mathrm{eq})}_{X(t)}(x_t) = \int_{-\infty}^{+\infty} \mathrm{d}x_s W(X(t) = x_t \leftarrow X(s) = x_s) p^{(\mathrm{eq})}_{X(s)}(x_s), \quad t > s; \tag{5.1.44}
$$

$$
(\beta)\ \int_{-\infty}^{+\infty} \mathrm{d}x_t W(X(t) = x_t \leftarrow X(s) = x_s) = 1, \quad \forall x_s \in \mathbb{R}, \quad t > s; \tag{5.1.45}
$$

$$
(\gamma)\ W(\cdot \leftarrow \cdot) \geq 0. \tag{5.1.46}
$$

We know from (5.1.41) that $(\beta)$ and $(\gamma)$ are self-evident. The statement $(\alpha)$ implies that $p^{(\mathrm{eq})}_{X(t)}(x_t)$ is a right eigenvector of the transition probability $W(\cdot \leftarrow \cdot)$, belonging to the eigenvalue 1.

In the theory of Markov processes, we are customarily given the initial state and the transition probability $W(\cdot \leftarrow \cdot)$ to obtain the stationary state and its probability density function $p^{(\mathrm{eq})}_{X(t)}(x_t)$. There also exists the reverse situation. Given the equilibrium probability density function $p^{(\mathrm{eq})}_{X(t)}(x_t)$, we attempt to obtain the transition probability $W(\cdot \leftarrow \cdot)$ which gives the probability density function $p^{(\mathrm{eq})}_{X(t)}(x_t)$ of the stationary state for an arbitrary initial state. Under these circumstances, we have the *detailed balance condition* as a sufficient condition to accomplish this goal:

$$(\delta)\quad \frac{W(X(t)=x_t \leftarrow X(s)=x_s)}{W(X(s)=x_s \leftarrow X(t)=x_t)} = \frac{p^{(\mathrm{eq})}_{X(t)}(x_t)}{p^{(\mathrm{eq})}_{X(s)}(x_s)}. \tag{5.1.47}$$

The point of this detailed balance condition, (5.1.47), is that if we choose $W$ satisfying ($\beta$), ($\gamma$) and ($\delta$), then the eigenvalue condition ($\alpha$) is automatically satisfied. It is self-evident that we find $p^{(\mathrm{eq})}_{X(t)}(x_t)$ from (5.1.43) if we generate $p^{(n)}_{X(t)}(x_t)$ by (5.1.42) with the use of this $W$ satisfying (5.1.47).

### 5.1.3 Evolution Equation of Quantum Mechanics and Quantum Field Theory

Finally comes the comparison of the composition law of the transition probability of a Markov process with that of the transition probability amplitude of quantum mechanics and quantum field theory.

For the transition probability of a Markov process, we have

$$\int_{-\infty}^{+\infty} \mathrm{d}x_u W(X(t)=x_t \leftarrow X(u)=x_u) W(X(u)=x_u \leftarrow X(s)=x_s) = W(X(t)=x_t \leftarrow X(s)=x_s), \tag{5.1.48}$$

where

$$t > u > s.$$

For the transition probability amplitude of quantum mechanics, we have

$$\int_{-\infty}^{+\infty} \mathrm{d}q_{t_{\mathrm{int}}}\ \langle q_{t_{\mathrm{f}}}, t_f | q_{t_{\mathrm{int}}}, t_{\mathrm{int}}\rangle \langle q_{t_{\mathrm{int}}}, t_{\mathrm{int}} | q_{t_i}, t_i\rangle = \langle q_{t_f}, t_f | q_{t_i}, t_i\rangle, \tag{5.1.49}$$

where

$$t_f > t_{\mathrm{int}} > t_i.$$

In (5.1.48), $t$, $s$ and $u$ are the evolution parameters of a Markov process, whereas in (5.1.49), $t_f$, $t_i$ and $t_{\mathrm{int}}$ are real time variables of quantum mechanics. Since we want to formulate quantum mechanics and quantum field theory as a stationary state of a Markov process, obviously the evolution parameter $\tau$ of the Markov process is missing in quantum mechanics and quantum field theory (i.e., the limit $\tau \to \infty$ has already been taken). Indeed, we cannot identify the real time variable $t$ of quantum theory as the evolution parameter of a Markov process. If we identify the real time $t$ as the evolution parameter of a Markov process, we do not obtain the Schrödinger equation. Thus, we shall introduce the evolution parameter $\tau$ into quantum theory. With the introduction of the evolution parameter $\tau$, we can regard the transition probability amplitudes of quantum mechanics and quantum field theory as a stochastic process.

In quantum mechanics, we replace

$$q_r(t)$$

by

$$q_r(t;\tau), \tag{5.1.50M}$$

and

$$I_{\mathrm{E}}[q] \equiv \int_{t_i}^{t_f} \mathrm{d}t L_{\mathrm{E}}\left(q_r(t), \frac{\mathrm{d}}{\mathrm{d}t}q_r(t)\right)$$

by

$$I_{\mathrm{E}}[q;\tau] \equiv \int_{t_i}^{t_f} \mathrm{d}t L_{\mathrm{E}}\left(q_r(t;\tau), \frac{\mathrm{d}}{\mathrm{d}t}q_r(t;\tau)\right). \tag{5.1.51M}$$

The evolution equation of quantum mechanics for a finite $\tau$ is given by

$$\frac{\partial}{\partial\tau}q_r(t;\tau) = \frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau)} + \eta_r(t;\tau). \tag{5.1.52M}$$

The stationary state conditions are given by

$$\lim_{\tau\to\infty} q_r(t;\tau) = q_r(t) \qquad \text{(a.s.)}, \tag{5.1.53M}$$

$$\lim_{\tau\to\infty} \frac{\partial}{\partial\tau}q_r(t;\tau) = 0 \qquad \text{(a.s.)}. \tag{5.1.54M}$$

The $\eta_r(t;\tau)$ is Gaussian white noise whose distribution functional $W[\eta]$ is given by

$$W[\eta] = \exp\left[-\frac{\beta}{4}\int \mathrm{d}t\mathrm{d}\tau \sum_{r=1}^{f} \eta_r^2(t;\tau)\right] \times\left\{\int \mathcal{D}[\eta]\exp\left[-\frac{\beta}{4}\int \mathrm{d}t\mathrm{d}\tau \sum_{r=1}^{f} \eta_r^2(t;\tau)\right]\right\}^{-1} \tag{5.1.55a}$$

where the white noise $\eta_r(t;\tau)$ satisfies

$$\lim_{\tau\to\infty} \eta_r(t;\tau) = \text{stationary Gaussian random variable.} \tag{5.1.55b}$$

When we designate the average with respect to $W[\eta]$ by $\langle\ \rangle$, we have

$$\langle \eta_r(t;\tau) \rangle = 0, \tag{5.1.55c}$$

and

$$\langle \eta_r(t;\tau)\eta_s(t';\tau') \rangle = \frac{2}{\beta}\delta_{rs}\delta(t-t')\delta(\tau-\tau'). \tag{5.1.55d}$$

From (5.1.52M)–(5.1.54M) and (5.1.55b), we have the stationary state equation of motion by taking the limit $\tau \to \infty$,

$$\frac{\delta I_{\mathrm{E}}[q]}{\delta q_r(t)} + \eta_r(t;\infty) = 0 \qquad \text{(a.s.)}, \tag{5.1.56M}$$

which is the Euler–Lagrange equation of motion for Euclidean quantum mechanics.

In quantum field theory, we replace

$$\phi_i(x)$$

by

$$\phi_i(x;\tau), \tag{5.1.50F}$$

and

$$I_{\mathrm{E}}[\phi] \equiv \int_{\Omega} \mathrm{d}^4x \mathcal{L}_{\mathrm{E}}(\phi_i(x), \partial_\mu \phi_i(x))$$

by

$$I_{\mathrm{E}}[\phi;\tau] \equiv \int_{\Omega} \mathrm{d}^4x \mathcal{L}_{\mathrm{E}}(\phi_i(x;\tau), \partial_\mu \phi_i(x;\tau)). \tag{5.1.51F}$$

The evolution equation of quantum field theory for finite $\tau$ is given by

$$\frac{\partial}{\partial \tau}\phi_i(x;\tau) = \frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta \phi_i(x;\tau)} + \tilde{\eta}_i(x;\tau). \tag{5.1.52F}$$

The stationary state conditions are given by

$$\lim_{\tau\to\infty} \phi_i(x;\tau) = \phi_i(x) \qquad \text{(a.s.)}, \tag{5.1.53F}$$

$$\lim_{\tau\to\infty} \frac{\partial}{\partial \tau}\phi_i(x;\tau) = 0 \qquad \text{(a.s.)}. \tag{5.1.54F}$$

The $\tilde{\eta}_i(x;\tau)$ is Gaussian white noise whose distribution functional $W[\tilde{\eta}]$ is given by

$$W[\tilde{\eta}] = \exp\left[-\frac{\beta}{4}\int \mathrm{d}^4x\mathrm{d}\tau \sum_{i=1}^{n} \tilde{\eta}_i^2(x;\tau)\right]$$
$$\times \left\{\int \mathcal{D}[\tilde{\eta}] \exp\left[-\frac{\beta}{4}\int \mathrm{d}^4x\mathrm{d}\tau \sum_{i=1}^{n} \tilde{\eta}_i^2(x;\tau)\right]\right\}^{-1} \qquad (5.1.55\mathrm{e})$$

where the white noise $\tilde{\eta}_i(x;\tau)$ satisfies

$$\lim_{\tau\to\infty} \tilde{\eta}_i(x;\tau) = \text{stationary Gaussian random variable.} \qquad (5.1.55\mathrm{f})$$

When we designate the average with respect to $W[\tilde{\eta}]$ by $\langle\ \rangle$, we have

$$\langle \tilde{\eta}_i(x;\tau)\rangle = 0, \qquad (5.1.55\mathrm{g})$$

and

$$\langle \tilde{\eta}_i(x;\tau)\tilde{\eta}_j(x';\tau')\rangle = \frac{2}{\beta}\delta_{ij}\delta^4(x-x')\delta(\tau-\tau'). \qquad (5.1.55\mathrm{h})$$

From (5.1.52F), (5.1.53F), (5.1.54F) and (5.1.55f), we obtain the stationary state equation of motion by taking the limit $\tau \to \infty$,

$$\frac{\delta I_{\mathrm{E}}[\phi]}{\delta\phi_i(x)} + \tilde{\eta}_i(x;\infty) = 0 \qquad \text{(a.s.)}, \qquad (5.1.56\mathrm{F})$$

which is the Euler–Lagrange equation of motion for Euclidean quantum field theory.

We have used the $c$-number notation in (5.1.50M) through (5.1.56F) and suppressed the variable $\omega \in \Omega$. We call (5.1.52M) and (5.1.52F) the *Parisi–Wu equations*.

The stochastic quantization method is the equation of motion approach to $c$-number quantization and, in principle, does not require the existence of the Lagrangian (density) as stated at the beginning of this chapter. Nevertheless, we assume the existence of the Lagrangian (density) and the action functional for the sake of simplicity of the presentation as we did in writing down the Parisi–Wu equations, (5.1.52M) and (5.1.52F). In the case when the original equation of motion is not derivable from the action functional, we merely replace

$$\frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau)} \quad \text{or} \quad \frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta\phi_i(x;\tau)} \qquad (5.1.57)$$

in the Parisi–Wu equation by the original equation of motion.

## 5.2 Stochastic Quantization of Non-Abelian Gauge Field

In this section, we first discuss the derivation of the Fokker–Planck equation from the Parisi–Wu equation. We write down the path integral representation for the transition probability. As the stationary solutions of the Fokker–Planck equation, we obtain quantum mechanics and quantum field theory in the Euclidean metric (Sect. 5.2.1). Secondly, we discuss the stochastic quantization of Abelian and non-Abelian gauge fields. We split the evolution equation for the gauge field into those of a transverse mode and a longitudinal mode. The former has a damping term and the initial distribution of the transverse mode does not persist in the correlation function in the limit $\tau = \tau' \to \infty$. The latter has no damping term and the initial distribution of the longitudinal mode persists in the correlation function in the limit $\tau = \tau' \to \infty$. The former provides the usual gluon contribution, while the latter provides the Faddeev–Popov ghost effect. The width of the initial symmetric distribution of the longitudinal mode is the gauge parameter of the covariant gauge "free" Green's function of the gauge field (Sect. 5.2.2). Thirdly, we discuss the covariant nonholonomic gauge-fixing condition, which cannot be written as an addition to the action functional of non-Abelian gauge fields. We find that the Lagrange multiplier $\xi$ is a covariant gauge parameter. We discuss the Fokker–Planck equation for non-Abelian gauge field theory with some modifications and show that non-Abelian gauge field theory can be seen as a stationary state of a stochastic process (Sect. 5.2.3).

### 5.2.1 Parisi–Wu Equation and Fokker–Planck Equation

Let us begin with the derivation of the *Fokker–Planck equation* from the *Parisi–Wu equation* via the path integral representation of the transition probability of a stochastic process.

For quantum mechanics and quantum field theory, we have the Parisi–Wu equations,

$$\frac{\partial}{\partial \tau} q_r(t;\tau) = \frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau)} + \eta_r(t;\tau), \tag{5.2.1M}$$

$$\frac{\partial}{\partial \tau} \phi_i(x;\tau) = \frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta \phi_i(x;\tau)} + \tilde{\eta}_i(x,\tau), \tag{5.2.1F}$$

$$\langle \eta_r(t;\tau) \rangle = 0,$$

$$\langle \eta_r(t;\tau)\eta_s(t';\tau') \rangle = \frac{2}{\beta}\delta_{rs}\delta(t-t')\delta(\tau-\tau'), \tag{5.2.2M}$$

$$\langle \tilde{\eta}_i(x;\tau) \rangle = 0,$$

$$\langle \tilde{\eta}_i(x;\tau)\tilde{\eta}_j(x',\tau') \rangle = \frac{2}{\beta}\delta_{ij}\delta^4(x-x')\delta(\tau-\tau'), \tag{5.2.2F}$$

where $\langle\ \rangle$ represents the average over the Gaussian distribution functional $W[\eta]$ for quantum mechanics,

$$W[\eta] = \exp\left[-\frac{\beta}{4}\int \mathrm{d}t\mathrm{d}\tau \sum_{r=1}^{f} \eta_r^2(t;\tau)\right] \times \left\{\int \mathcal{D}[\eta]\exp\left[-\frac{\beta}{4}\int \mathrm{d}t\mathrm{d}\tau \sum_{r=1}^{f} \eta_r^2(t;\tau)\right]\right\}^{-1}, \tag{5.2.3M}$$

and the average over the Gaussian distribution functional $W[\tilde{\eta}]$ for quantum field theory,

$$W[\tilde{\eta}] = \exp\left[-\frac{\beta}{4}\int \mathrm{d}^4x\mathrm{d}\tau \sum_{k=1}^{n} \tilde{\eta}_k^2(x;\tau)\right] \times \left\{\int \mathcal{D}[\tilde{\eta}]\exp\left[-\frac{\beta}{4}\int \mathrm{d}^4x\mathrm{d}\tau \sum_{k=1}^{n} \tilde{\eta}_k^2(x;\tau)\right]\right\}^{-1}. \tag{5.2.3F}$$

The presence of $\delta(\tau-\tau')$ in (5.2.2M) and (5.2.2F) suggests that the stochastic process under consideration is a Markov process. We write the transition probability of the stochastic process for $\{q_r(t;\tau_i)\}_{r=1}^{f} \to \{q_r(t;\tau_f)\}_{r=1}^{f}$ as

$$T(q_f,\tau_f \mid q_i,\tau_i) = \lim_{N\to\infty}\int\cdots\int \prod_{j=1}^{N-1}\{\mathrm{d}q_{(j)}T(q_{(j+1)},\tau_{j+1} \mid q_{(j)},\tau_j)\} \times T(q_{(1)},\tau_1 \mid q_i,\tau_i), \tag{5.2.4M}$$

where

$$q_{(j)} = \{q_r(t;\tau_j)\}_{r=1}^{f}, \quad q_{(0)} \equiv q_i = \{q_r(t;\tau_i)\}_{r=1}^{f},$$

$$q_{(N)} \equiv q_f = \{q_r(t;\tau_f)\}_{r=1}^{f},$$

$$\tau_j = j\Delta\tau, \quad j=1,\ldots,N, \quad \Delta\tau = \frac{\tau_f-\tau_i}{N}, \quad \mathrm{d}q_{(j)} = \prod_{r=1}^{f}\mathrm{d}q_{r,j}, \tag{5.2.5M}$$

and likewise for $\{\phi_k(x;\tau_i)\}_{k=1}^n \to \{\phi_k(x;\tau_f)\}_{k=1}^n$ as

$$T(\phi_f,\tau_f \mid \phi_i,\tau_i) = \lim_{N\to\infty} \int \cdots \int \prod_{j=1}^{N-1} \{\mathrm{d}\phi_{(j)} T(\phi_{(j+1)},\tau_{j+1} \mid \phi_{(j)},\tau_j)\} \times T(\phi_{(1)},\tau_1 \mid \phi_i,\tau_i), \tag{5.2.4F}$$

where

$$\phi_{(j)} = \{\phi_k(x;\tau_j)\}_{k=1}^n, \quad \phi_{(0)} \equiv \phi_i = \{\phi_k(x;\tau_i)\}_{k=1}^n,$$

$$\phi_{(N)} \equiv \phi_f = \{\phi_k(x;\tau_f)\}_{k=1}^n,$$

$$\tau_j = j\Delta\tau, \quad j = 1,\ldots,N, \quad \Delta\tau = \frac{\tau_f - \tau_i}{N}, \quad d\phi_{(j)} = \prod_{k=1}^n \mathrm{d}\phi_{k,j}. \tag{5.2.5F}$$

Setting

$$\xi_{r,j}(t) \equiv \int_{\tau_j}^{\tau_{j+1}} \mathrm{d}\tau \eta_r(t;\tau), \tag{5.2.6M}$$

$$\tilde{\xi}_{k,j}(x) \equiv \int_{\tau_j}^{\tau_{j+1}} \mathrm{d}\tau \tilde{\eta}_k(x;\tau), \tag{5.2.6F}$$

we have

$$\langle \xi_{r,j}(t) \rangle = \int_{\tau_j}^{\tau_{j+1}} \mathrm{d}\tau \langle \eta_r(t;\tau) \rangle = 0, \tag{5.2.7M}$$

$$\begin{aligned}\langle \xi_{r,j}(t), \xi_{s,j}(t') \rangle &= \int_{\tau_j}^{\tau_{j+1}} \mathrm{d}\tau \int_{\tau_j}^{\tau_{j+1}} \mathrm{d}\tau' \langle \eta_r(t;\tau)\eta_s(t';\tau') \rangle \\ &= \frac{2}{\beta}\delta_{rs}\delta(t-t')\Delta\tau,\end{aligned} \tag{5.2.8M}$$

$$\langle \tilde{\xi}_{k,j}(x) \rangle = \int_{\tau_j}^{\tau_{j+1}} \mathrm{d}\tau \langle \tilde{\eta}_k(x;\tau) \rangle = 0, \tag{5.2.7F}$$

$$\begin{aligned}\langle \tilde{\xi}_{k,j}(x)\tilde{\xi}_{l,j}(x') \rangle &= \int_{\tau_j}^{\tau_{j+1}} \mathrm{d}\tau \int_{\tau_j}^{\tau_{j+1}} \mathrm{d}\tau' \langle \tilde{\eta}_k(x,\tau)\tilde{\eta}_l(x',\tau') \rangle \\ &= \frac{2}{\beta}\delta_{kl}\delta^4(x-x')\Delta\tau.\end{aligned} \tag{5.2.8F}$$

From the Parisi–Wu equations, (5.2.1M) and (5.2.1F), we can set

$$T(q_{(j)},\tau_j \mid q_{(j-1)},\tau_{j-1}) = W_\xi(x_j)\mathrm{d}x_j = C\exp\left[-\frac{\beta}{2\Delta\tau}\sum_{r=1}^{f}\frac{1}{2}x_{r,j}^2\right]\mathrm{d}x_j, \tag{5.2.9M}$$

which is the probability of the Gaussian random variable $\xi_{r,j}(t)$ taking a value in the interval $\{[x_{r,j}+\mathrm{d}x_{r,j}, x_{r,j}]\}_{r=1}^{f}$ with $x_{r,j}$ specified by

$$x_{r,j} \equiv \Delta\tau\cdot\eta_r(t;\tau_j) \equiv \Delta\tau\left\{\frac{q_r(t;\tau_{j+1})-q_r(t;\tau_j)}{\Delta\tau} - \frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau_j)}\right\}, \tag{5.2.10M}$$

and

$$T(\phi_{(j)},\tau_j \mid \phi_{(j-1)},\tau_{j-1}) = W_\xi(\tilde{x}_j)\mathrm{d}\tilde{x}_j = C\exp\left[-\frac{\beta}{2\Delta\tau}\sum_{k=1}^{n}\frac{1}{2}\tilde{x}_{k,j}^2\right]\mathrm{d}\tilde{x}_j, \tag{5.2.9F}$$

which is the probability of the Gaussian random variable $\tilde{\xi}_{k,j}(x)$ taking a value in the interval $\{[\tilde{x}_{k,j}+\mathrm{d}\tilde{x}_{k,j}, \tilde{x}_{k,j}]\}_{k=1}^{n}$ with $\tilde{x}_{k,j}$ specified by

$$\tilde{x}_{k,j} \equiv \Delta\tau\cdot\tilde{\eta}_k(x;\tau_j) \equiv \Delta\tau\left\{\frac{\phi_k(x;\tau_{j+1})-\phi_k(x;\tau_j)}{\Delta\tau} - \frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta \phi_k(x;\tau_j)}\right\}. \tag{5.2.10F}$$

Thus, we have for quantum mechanics

$$\begin{aligned} T(q_{(j)},\tau_j \mid q_{(j-1)},\tau_{j-1}) \\ = C\exp\left[-\frac{\beta}{4}\sum_{r=1}^{f}\Delta\tau\left\{\frac{q_{r,j+1}-q_{r,j}}{\Delta\tau} - \frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau_j)}\right\}^2\right], \end{aligned} \tag{5.2.11M}$$

and for quantum field theory

$$\begin{aligned} T(\phi_{(j)},\tau_j \mid \phi_{(j-1)},\tau_{j-1}) \\ = C\exp\left[-\frac{\beta}{4}\sum_{k=1}^{n}\Delta\tau\left\{\frac{\phi_{k,j+1}-\phi_{k,j}}{\Delta\tau} - \frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta \phi_k(x;\tau_j)}\right\}^2\right]. \end{aligned} \tag{5.2.11F}$$

Substituting Eq(5.2.11M) into (5.2.4M), we have

$$\begin{aligned} T(q_f,\tau_f \mid q_i,\tau_i) = \lim_{N\to\infty}\int\cdots\int\prod_{j=1}^{N-1}\{\mathrm{d}q_{(j)}T(q_{(j+1)},\tau_{j+1} \mid q_{(j)},\tau_j)\} \\ \times T(q_{(1)},\tau_1 \mid q_i,\tau_i) \end{aligned}$$

$$= C' \int_{q(t;\tau_i)=q_i}^{q(t;\tau_f)=q_f} \mathcal{D}[q(t;\tau)] \exp\left[-\frac{\beta}{2}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \Lambda\left(q(\tau), \frac{\partial}{\partial\tau}q(\tau), \tau\right)\right], \tag{5.2.12M}$$

where $C'$ is a normalization constant and $\Lambda$ is given by

$$\Lambda\left(q(\tau), \frac{\partial}{\partial\tau}q(\tau), \tau\right) \equiv \frac{1}{2}\int \mathrm{d}t \sum_{r=1}^{f}\left\{\frac{\partial q_r(t;\tau)}{\partial\tau} - \frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau)}\right\}^2. \tag{5.2.13M}$$

Substituting (5.2.11F) into (5.2.4F), we have

$$T(\phi_f, \tau_f \mid \phi_i, \tau_i) = \lim_{N\to\infty}\int\cdots\int \prod_{j=1}^{N-1}\{\mathrm{d}\phi_{(j)} T(\phi_{(j+1)}, \tau_{j+1} \mid \phi_{(j)}, \tau_j)\}$$
$$\times T(\phi_{(1)}, \tau_1 \mid \phi_i, \tau_i)$$

$$= C'' \int_{\phi(x;\tau_i)=\phi_i}^{\phi(x;\tau_f)=\phi_f} \mathcal{D}[\phi(x;\tau)] \exp\left[-\frac{\beta}{2}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \tilde{\Lambda}\left(\phi(\tau), \frac{\partial}{\partial\tau}\phi(\tau), \tau\right)\right], \tag{5.2.12F}$$

where $C''$ is a normalization constant and $\tilde{\Lambda}$ is given by

$$\tilde{\Lambda}\left(\phi(\tau), \frac{\partial}{\partial\tau}\phi(\tau), \tau\right) \equiv \frac{1}{2}\int \mathrm{d}^4x \sum_{k=1}^{n}\left\{\frac{\partial \phi_k(x;\tau)}{\partial\tau} - \frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta \phi_k(x;\tau)}\right\}^2. \tag{5.2.13F}$$

Equations (5.2.12M and F) and (5.2.13M and F) suggest the following correspondence,

$$-\frac{2}{\beta} \leftrightarrow \frac{\hbar}{\mathrm{i}}, \tag{5.2.14}$$

$$\Lambda \text{ and } \tilde{\Lambda} \leftrightarrow \text{Lagrangian}, \tag{5.2.15}$$

between the theory of stochastic processes and quantum theory.

We derive the Fokker–Planck equation from (5.2.12M) for quantum mechanics:

$$\frac{2}{\beta}\frac{\partial}{\partial\tau_f}T(q_f, \tau_f \mid q_i, \tau_i) = \mathcal{F}\left(-\frac{2}{\beta}\frac{\partial}{\partial q_f}, q_f, \tau_f\right) T(q_f, \tau_f \mid q_i, \tau_i), \tag{5.2.16M}$$

where the Fokker–Planck operator

$$\mathcal{F}\left(-\frac{2}{\beta}\frac{\partial}{\partial q_{\mathrm{f}}}, q_{\mathrm{f}}, \tau_{\mathrm{f}}\right)$$

plays the role of the Hamiltonian operator, which is obtained from the "Hamiltonian" $\mathcal{F}(p, q, \tau)$ by replacement of

$$p_r \text{ by } -\frac{2}{\beta}\frac{\partial}{\partial q_r}. \tag{5.2.17M}$$

We define the "Hamiltonian" $\mathcal{F}(p, q, \tau)$ as the Legendre transform of the "Lagrangian" $\Lambda(q(\tau), \partial q(\tau)/\partial\tau, \tau)$,

$$\mathcal{F}(p, q, \tau) \equiv \int \mathrm{d}t \sum_{r=1}^{f} p_r(t;\tau)\frac{\partial}{\partial\tau}q_r(t;\tau) - \Lambda\left(q_s(\tau), \frac{\partial}{\partial\tau}q_s(\tau), \tau\right), \tag{5.2.18M}$$

where the "momentum" $p_r(t;\tau)$ conjugate to $q_r(t;\tau)$ is defined by

$$p_r(t;\tau) \equiv \frac{\delta\Lambda\left(q(\tau), \frac{\partial q(\tau)}{\partial\tau}, \tau\right)}{\delta\left(\frac{\partial q_r(t;\tau)}{\partial\tau}\right)}. \tag{5.2.19M}$$

We assume that (5.2.19M) is solvable for $\partial q_s(t;\tau)/\partial\tau$ in terms of $q$ and $p$. The "Lagrangian", (5.2.13M), provides us with the "momentum" $p_r(t;\tau)$ as

$$p_r(t;\tau) = \frac{\partial}{\partial\tau}q_r(t;\tau) - \frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau)}, \tag{5.2.20M}$$

and the "Hamiltonian" $\mathcal{F}(p, q, \tau)$ as

$$\mathcal{F}(p, q, \tau) = \sum_{r=1}^{f}\int \mathrm{d}t \left\{\frac{1}{2}p_r^2(t;\tau) + p_r(t;\tau)\frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau)}\right\}. \tag{5.2.21M}$$

After the replacement of

$$p_r(t;\tau) \quad \text{by} \quad -\frac{2}{\beta}\frac{\delta}{\delta q_r(t;\tau)}, \tag{5.2.22M}$$

we have the Fokker–Planck equation

$$\frac{\partial}{\partial\tau}\psi[q,\tau] = \sum_{r=1}^{f}\int \mathrm{d}t \frac{\delta}{\delta q_r(t;\tau)}\left\{\frac{1}{\beta}\frac{\delta}{\delta q_r(t;\tau)} - \frac{\delta I_{\mathrm{E}}[q;\tau]}{\delta q_r(t;\tau)}\right\}\psi[q,\tau], \tag{5.2.23M}$$

for the probability distribution functional $\psi[q(\tau),\tau]$ of the stochastic process governed by the Parisi–Wu equation, (5.2.1M), (5.2.2M) and (5.2.3M). The

Fokker–Planck equation (5.2.33M) for quantum mechanics has the stationary solution

$$\psi^{(\mathrm{eq})}[q(t)] = \frac{\exp[\beta I_{\mathrm{E}}[q]]}{\int \mathcal{D}[q] \exp\left[\beta I_{\mathrm{E}}[q]\right]}. \tag{5.2.24M}$$

We derive the Fokker–Planck equation from (5.2.12F) for quantum field theory:

$$\frac{2}{\beta}\frac{\partial}{\partial \tau_f} T(\phi_f, \tau_f \mid \phi_i, \tau_i) = \tilde{\mathcal{F}}\left(-\frac{2}{\beta}\frac{\delta}{\delta \phi_f}, \phi_f, \tau_f\right) T(\phi_f, \tau_f \mid \phi_i, \tau_i), \tag{5.2.16F}$$

where the Fokker–Planck operator

$$\tilde{\mathcal{F}}\left(-\frac{2}{\beta}\frac{\delta}{\delta \phi_f}, \phi_f, \tau_f\right)$$

plays the role of the Hamiltonian operator, which is obtained from the "Hamiltonian" $\tilde{\mathcal{F}}(\pi, \phi, \tau)$ by the replacement of

$$\pi_k \text{ by } -\frac{2}{\beta}\frac{\delta}{\delta \phi_k}. \tag{5.2.17F}$$

We define the "Hamiltonian" $\tilde{\mathcal{F}}(\pi, \phi, \tau)$ as the Legendre transform of the "Lagrangian" $\tilde{\Lambda}(\phi_l(\tau), \frac{\partial \phi_l(\tau)}{\partial \tau}, \tau)$,

$$\tilde{\mathcal{F}}(\pi, \phi, \tau) = \sum_{k=1}^{n} \int \mathrm{d}^4 x \pi_k(x;\tau) \frac{\partial}{\partial \tau}\phi_k(x;\tau) - \tilde{\Lambda}\left(\phi_l(\tau), \frac{\partial}{\partial \tau}\phi_l(\tau), \tau\right), \tag{5.2.18F}$$

where the "momentum" $\pi_k(x;\tau)$ conjugate to $\phi_k(x;\tau)$ is defined by

$$\pi_k(x;\tau) \equiv \frac{\delta \tilde{\Lambda}\left(\phi_l(\tau), \frac{\partial \phi_l(\tau)}{\partial \tau}, \tau\right)}{\delta\left(\frac{\partial \phi_k(x;\tau)}{\partial \tau}\right)}. \tag{5.2.19F}$$

We assume that (5.2.19F) is solvable for $\frac{\partial \phi_l(x;\tau)}{\partial \tau}$ in terms of $\phi$ and $\pi$. The "Lagrangian", (5.2.13F), provides us with the "momentum" $\pi_k(x;\tau)$ as

$$\pi_k(x;\tau) = \frac{\partial}{\partial \tau}\phi_k(x;\tau) - \frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta \phi_k(x;\tau)}, \tag{5.2.20F}$$

and the Hamiltonian $\tilde{\mathcal{F}}(\pi, \phi, \tau)$ as

$$\tilde{\mathcal{F}}(\pi, \phi, \tau) = \sum_{k=1}^{n} \int \mathrm{d}^4 x \left\{\frac{1}{2}\pi_k^2(x;\tau) + \pi_k(x;\tau)\frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta \phi_k(x;\tau)}\right\}. \tag{5.2.21F}$$

After the replacement of

$$\pi_k(x;\tau) \text{ by } -\frac{2}{\beta}\frac{\delta}{\delta\phi_k(x;\tau)}, \tag{5.2.22F}$$

we have the Fokker–Planck equation

$$\frac{\partial}{\partial\tau}\psi[\phi;\tau] = \sum_{k=1}^{n}\int \mathrm{d}^4x \frac{\delta}{\delta\phi_k(x;\tau)}\left\{\frac{1}{\beta}\frac{\delta}{\delta\phi_k(x;\tau)} - \frac{\delta I_{\mathrm{E}}[\phi;\tau]}{\delta\phi_k(x;\tau)}\right\}\psi[\phi;\tau], \tag{5.2.23F}$$

for the probability distribution functional $\psi[\phi;\tau]$ of the stochastic process governed by the Parisi–Wu equation, (5.2.1F)–(5.2.3F). The Fokker–Planck equation (5.2.23F) for the quantum field theory of a nonsingular system has the stationary solution,

$$\psi^{(\mathrm{eq})}[\phi] = \frac{\exp[\beta I_{\mathrm{E}}[\phi]]}{\int \mathcal{D}[\phi]\exp[\beta I_{\mathrm{E}}[\phi]]}. \tag{5.2.24F}$$

In view of (5.1.56M), (5.1.56F), (5.2.24M) and (5.2.24F), we succeeded in formulating Euclidean quantum mechanics and Euclidean quantum field theory of a nonsingular system as a stationary state of a stochastic process. This is the basic idea of the method of stochastic quantization of Parisi and Wu. Stochastic quantization is a $c$-number quantization just like path integral quantization. Later in Sect. 5.2.3, we shall obtain the stationary solution of the Fokker–Planck equation for non-Abelian gauge field theory, with an appropriate modification to the Fokker–Planck equation.

### 5.2.2 Stochastic Quantization of Abelian and Non-Abelian Gauge Fields

In the first place, we discuss the stochastic quantization of the Abelian gauge field $A_\mu(x)$. The Parisi–Wu equation for Abelian gauge fields is given by

$$\frac{\partial}{\partial\tau}A_\mu(x;\tau) = (\delta_{\mu\nu}\partial^2 - \partial\hat{\mu}\partial_\nu)A_\nu(x;\tau) + \eta_\mu(x;\tau), \tag{5.2.25C}$$

$$\langle\eta_\mu(x;\tau)\rangle = 0,$$

$$\langle\eta_\mu(x;\tau)\eta_\nu(x';\tau')\rangle = 2\delta_{\mu\nu}\delta^4(x-x')\delta(\tau-\tau'). \tag{5.2.26C}$$

In terms of the Fourier component with respect to $x$, we have

$$\frac{\partial}{\partial\tau}\tilde{A}_\mu(k;\tau) = -k^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\tilde{A}_\nu(k;\tau) + \tilde{\eta}_\mu(k;\tau), \tag{5.2.25F}$$

$$\langle \tilde{\eta}_\mu(k;\tau) \rangle = 0,$$

$$\langle \tilde{\eta}_\mu(k;\tau)\tilde{\eta}_\nu(k';\tau') \rangle = 2\delta_{\mu\nu}\delta^4(k+k')\delta(\tau-\tau'). \tag{5.2.26F}$$

We consider the transverse mode equation and the longitudinal mode equation separately. We split $\tilde{A}_\mu(k;\tau)$ ($\tilde{\eta}_\mu(k;\tau)$) into the transverse mode $\tilde{A}^{\mathrm{T}}_\mu(k;\tau)$ ($\tilde{\eta}^{\mathrm{T}}_\mu(k;\tau)$) and the longitudinal mode $\tilde{A}^{\mathrm{L}}_\mu(k;\tau)$ ($\tilde{\eta}^{\mathrm{L}}_\mu(k;\tau)$), i.e.,

$$\tilde{A}^{\mathrm{T}}_\mu(k;\tau) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\tilde{A}_\nu(k;\tau), \tag{5.2.27}$$

$$\tilde{A}^{\mathrm{L}}_\mu(k;\tau) = \frac{k_\mu k_\nu}{k^2}\tilde{A}_\nu(k;\tau), \tag{5.2.28}$$

$$\tilde{\eta}^{\mathrm{T}}_\mu(k;\tau) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\tilde{\eta}_\nu(k;\tau), \tag{5.2.29}$$

$$\tilde{\eta}^{\mathrm{L}}_\mu(k;\tau) = \frac{k_\mu k_\nu}{k^2}\tilde{\eta}_\nu(k;\tau). \tag{5.2.30}$$

We split the Parisi–Wu equation, (5.2.25F), into a transverse mode and a longitudinal mode

$$\frac{\partial}{\partial\tau}\tilde{A}^{\mathrm{T}}_\mu(k;\tau) = -k^2\tilde{A}^{\mathrm{T}}_\mu(k;\tau) + \tilde{\eta}^{\mathrm{T}}_\mu(k;\tau), \tag{5.2.31}$$

$$\frac{\partial}{\partial\tau}\tilde{A}^{\mathrm{L}}_\mu(k;\tau) = \tilde{\eta}^{\mathrm{L}}_\mu(k;\tau). \tag{5.2.32}$$

We observe that the evolution equation of the transverse mode, (5.2.31), has a damping term, $-k^2\tilde{A}^{\mathrm{T}}_\mu(k;\tau)$, while that of the longitudinal mode, (5.2.32), does not have a damping term due to gauge invariance. This implies that the initial distribution of the transverse mode does not persist in the limit $\tau \to \infty$, while the initial distribution of the longitudinal mode persists in the limit $\tau \to \infty$. We let the initial distribution of the longitudinal mode of an Abelian gauge field at $\tau = 0$ be given by

$$\tilde{A}^{\mathrm{L}}_\mu(k;0) = \frac{k_\mu}{k^2}\phi(k). \tag{5.2.33}$$

We solve (5.2.31) and (5.2.32), obtaining

$$\tilde{A}^{\mathrm{T}}_\mu(k;\tau) = \exp[-k^2\tau]\int_0^\tau \mathrm{d}\tau' \exp[k^2\tau']\tilde{\eta}^{\mathrm{T}}_\mu(k;\tau'), \tag{5.2.34}$$

$$\tilde{A}^{\mathrm{L}}_\mu(k;\tau) = \frac{k_\mu}{k^2}\phi(k) + \int_0^\tau \mathrm{d}\tau'' \tilde{\eta}^{\mathrm{L}}_\mu(k;\tau''), \tag{5.2.35}$$

where the initial distribution of the transverse mode $\tilde{A}^{\mathrm{T}}_\mu(k;\tau)$ is dropped in (5.2.34) from the beginning, anticipating the future limit $\tau \to \infty$. We calculate the correlation function of the Abelian gauge field:

$$\begin{aligned}&\langle \tilde{A}_\mu(k;\tau)\tilde{A}_\nu(k';\tau')\rangle \\ &\quad= \delta^4(k+k')\frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\{\exp[k^2(\tau-\tau')] - \exp[-k^2(\tau+\tau')]\} \\ &\qquad+ \frac{k_\mu k'_\nu}{k^2 k'^2}\phi(k)\phi(k') + 2\tau\delta^4(k+k')\frac{k_\mu k_\nu}{k^2}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\lim_{\tau=\tau'\to\infty}\langle \tilde{A}_\mu(k;\tau)\tilde{A}_\nu(k';\tau')\rangle \\ &\quad= \delta^4(k+k')\left\{\frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + 2\tau\frac{k_\mu k_\nu}{k^2}\right\} + \frac{k_\mu k'_\nu}{k^2 k'^2}\phi(k)\phi(k').\end{aligned} \tag{5.2.36}$$

If we set

$$\phi(k) = 0 \quad \text{(a.s.)}, \tag{5.2.37}$$

we get the original result obtained by Parisi and Wu for Abelian gauge fields:

$$\lim_{\tau=\tau'\to\infty}\langle \tilde{A}_\mu(k;\tau)\tilde{A}_\nu(k';\tau')\rangle = \delta^4(k+k')\left\{\frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + 2\tau\frac{k_\mu k_\nu}{k^2}\right\}. \tag{5.2.38}$$

i.e., the Landau gauge "free" Green's function plus a divergent term proportional to $\tau$. A distribution like (5.2.37), however, is rather exceptional. We assume that the initial distribution $\phi(k)$ is symmetric around $k = 0$, in order to restore the homogeneity of space-time, and we take the average of the last term in (5.2.36) with respect to $\phi(k)$. We find that the average of $\phi(k)\phi(k')$ is given by

$$\langle\!\langle \phi(k)\phi(k') \rangle\!\rangle^\phi = -\alpha\delta^4(k+k'), \tag{5.2.39}$$

where $\alpha$ is the width of the distribution of $\phi(k)$. From (5.2.36) and (5.2.39), we obtain

$$\begin{aligned}&\lim_{\tau=\tau'\to\infty}\langle \tilde{A}_\mu(k;\tau)\tilde{A}_\nu(k';\tau')\rangle \\ &\quad= \delta^4(k+k')\left\{\frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) + 2\tau\frac{k_\mu k_\nu}{k^2}\right\},\end{aligned} \tag{5.2.40}$$

i.e., the covariant gauge "free" Green's function plus a divergent term proportional to $\tau$. The width $\alpha$ of the initial distribution of the longitudinal mode $\phi(k)$ is a covariant gauge parameter.

In the second place, we discuss the stochastic quantization of non-Abelian gauge field theory. The Parisi–Wu equation for non-Abelian gauge field is of the following form

$$\frac{\partial}{\partial\tau}A_{\alpha\mu}(x;\tau) = (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A_{\alpha\mu}(x;\tau) \\ +(C_2AA + C_3AAA)_{\alpha\mu} + \eta_{\alpha\mu}(x;\tau), \tag{5.2.41C}$$

where

$$\langle\eta_{\alpha\mu}(x;\tau)\rangle = 0, \\ \langle\eta_{\alpha\mu}(x;\tau)\eta_{\beta\nu}(x';\tau')\rangle = 2\delta_{\alpha\beta}\delta_{\mu\nu}\delta^4(x-x')\delta(\tau-\tau'). \tag{5.2.42C}$$

The Fourier component of (5.2.41C) with respect to $x$ takes the form

$$\frac{\partial}{\partial\tau}\tilde{A}_{\alpha\mu}(k;\tau) = -k^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\tilde{A}_{\alpha\mu}(k;\tau) \\ +(C_2\tilde{A}\tilde{A} + C_3\tilde{A}\tilde{A}\tilde{A})_{\alpha\mu} + \tilde{\eta}_{\alpha\mu}(k;\tau), \tag{5.2.41F}$$

where

$$\langle\tilde{\eta}_{\alpha\mu}(k;\tau)\rangle = 0, \\ \langle\tilde{\eta}_{\alpha\mu}(k;\tau)\tilde{\eta}_{\beta\nu}(k';\tau')\rangle = 2\delta_{\alpha\beta}\delta_{\mu\nu}\delta^4(k+k')\delta(\tau-\tau'). \tag{5.2.42F}$$

As before, we split the Parisi–Wu equation (5.2.41F) into a transverse mode and a longitudinal mode:

$$\frac{\partial}{\partial\tau}\tilde{A}^{\mathrm{T}}_{\alpha\mu}(k;\tau) = -k^2\tilde{A}^{\mathrm{T}}_{\alpha\mu}(k;\tau) + (C_2\tilde{A}\tilde{A} + C_3\tilde{A}\tilde{A}\tilde{A})^{\mathrm{T}}_{\alpha\mu} + \tilde{\eta}^{\mathrm{T}}_{\alpha\mu}(k;\tau), \tag{5.2.43}$$

$$\frac{\partial}{\partial\tau}\tilde{A}^{\mathrm{L}}_{\alpha\mu}(k;\tau) = (C_2\tilde{A}\tilde{A} + C_3\tilde{A}\tilde{A}\tilde{A})^{\mathrm{L}}_{\alpha\mu} + \tilde{\eta}^{\mathrm{L}}_{\alpha\mu}(k;\tau). \tag{5.2.44}$$

In order to obtain the "free" Green's function of the non-Abelian gauge field, the $C_2$ terms and $C_3$ terms representing the self-interactions in (5.2.43) and (5.2.44) are immaterial. For the sake of obtaining the "free" Green's function, we shall write

$$\frac{\partial}{\partial\tau}\tilde{A}^{\mathrm{T}}_{\alpha\mu}(k;\tau) = -k^2\tilde{A}^{\mathrm{T}}_{\alpha\mu}(k;\tau) + \tilde{\eta}^{\mathrm{T}}_{\alpha\mu}(k;\tau), \tag{5.2.45}$$

$$\frac{\partial}{\partial \tau}\tilde{A}^{\mathrm{L}}_{\alpha\mu}(k;\tau) = \tilde{\eta}^{\mathrm{L}}_{\alpha\mu}(k;\tau), \tag{5.2.46}$$

as in the case of an Abelian gauge field, (5.2.31) and (5.2.32). The evolution equation for the transverse mode, (5.2.45), has a damping term $-k^2\tilde{A}^{\mathrm{T}}_{\alpha\mu}(k;\tau)$, and hence the initial distribution of $A^{\mathrm{T}}_{\alpha\mu}(k;\tau)$ does not persist in the limit $\tau \to \infty$, while the evolution equation for the longitudinal mode, (5.2.46), does not have a damping term, and the initial distribution of $A^{\mathrm{L}}_{\alpha\mu}(k;\tau)$ persists in the limit $\tau \to \infty$. We set the initial distribution of the longitudinal mode to be given by

$$\tilde{A}^{\mathrm{L}}_{\alpha\mu}(k;0) = \frac{k_\mu}{k^2}\phi_\alpha(k). \tag{5.2.47}$$

Solving (5.2.45) and (5.2.46),

$$\tilde{A}^{\mathrm{T}}_{\alpha\mu}(k;\tau) = \exp[-k^2\tau]\int_0^\tau \mathrm{d}\tau' \exp[k^2\tau']\tilde{\eta}^{\mathrm{T}}_{\alpha\mu}(k;\tau'), \tag{5.2.48}$$

$$\tilde{A}^{\mathrm{L}}_{\alpha\mu}(k;\tau) = \frac{k_\mu}{k^2}\phi_\alpha(k) + \int_0^\tau \mathrm{d}\tau''\tilde{\eta}^{\mathrm{L}}_{\alpha\mu}(k;\tau''), \tag{5.2.49}$$

where the initial distribution of the transverse mode is dropped in (5.2.48) from the beginning, anticipating the future limit $\tau \to \infty$. We calculate the correlation function of the non-Abelian gauge field:

$$\begin{aligned} &\langle \tilde{A}_{\alpha\mu}(k;\tau)\tilde{A}_{\beta\nu}(k';\tau')\rangle \\ &= \delta_{\alpha\beta}\delta^4(k+k')\frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\{\exp[k^2(\tau-\tau')] - \exp[-k^2(\tau+\tau')]\} \\ &\quad + \frac{k_\mu k'_\nu}{k^2 k'^2}\phi_\alpha(k)\phi_\beta(k') + 2\tau\delta_{\alpha\beta}\delta^4(k+k')\frac{k_\mu k_\nu}{k^2}. \end{aligned} \tag{5.2.50}$$

Thus, we have

$$\begin{aligned} &\lim_{\tau=\tau'\to\infty}\langle \tilde{A}_{\alpha\mu}(k;\tau)\tilde{A}_{\beta\nu}(k';\tau')\rangle \\ &= \delta_{\alpha\beta}\delta^4(k+k')\left\{\frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + 2\tau\frac{k_\mu k_\nu}{k^2}\right\} + \frac{k_\mu k'_\nu}{k^2 k'^2}\phi_\alpha(k)\phi_\beta(k'). \end{aligned} \tag{5.2.51}$$

In order to restore the homogeneity of space-time, as in the Abelian gauge field case, we assume that the initial distribution of the longitudinal mode $\phi_\alpha(k)$ is symmetric about $k=0$ with width $\xi$. We average the last term in (5.2.51) with respect to $\phi_\alpha(k)$. We find that the average of $\phi_\alpha(k)\phi_\beta(k')$ is given by

$$\langle\!\langle \phi_\alpha(k)\phi_\beta(k')\rangle\!\rangle^\phi = -\xi\delta_{\alpha\beta}\delta^4(k+k'). \tag{5.2.52}$$

From (5.2.51) and (5.2.52), we have

$$\lim_{\tau=\tau'\to\infty}\langle\tilde{A}_{\alpha\mu}(k;\tau)\tilde{A}_{\beta\nu}(k';\tau')\rangle$$
$$=\delta_{\alpha\beta}\delta^4(k+k')\left\{\frac{1}{k^2}\left(\delta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)+2\tau\frac{k_\mu k_\nu}{k^2}\right\}, \qquad (5.2.53)$$

i.e., the covariant gauge "free" Green's function plus a divergent term proportional to $\tau$. The width $\xi$ of the initial distribution of the longitudinal mode $\phi_\alpha(k)$ is a covariant gauge parameter.

According to an extensive calculation of M. Namiki et al., the $C_2$ term and $C_3$ term in the transverse equation, (5.2.43), provide the usual gluon contribution, while the $C_2$ term and $C_3$ term in the longitudinal equation, (5.2.44), provide both the Faddeev–Popov ghost effect and a divergent term proportional to $\tau$ in the gauge field "free" Green's function.

For both Abelian gauge field theory and non-Abelian gauge field theory, we obtained covariant gauge "free" Green's functions. The gauge parameter is the width of the initial distribution of the longitudinal mode in both cases. We obtained these results without the gauge-fixing term at the outset. At the same time, however, we have a divergent term proportional to $\tau$ in the gauge field "free" Green's function.

### 5.2.3 Covariant Nonholonomic Gauge-Fixing Condition and Stochastic Quantization of the Non-Abelian Gauge Field

In order to eliminate both the initial distribution term of the longitudinal mode and the divergent term proportional to $\tau$ from the gauge field "free" Green's function, M. Namiki et al. proposed a covariant nonholonomic gauge-fixing condition of the form

$$\frac{1}{\xi}\int \mathrm{d}^4x\delta A_{\alpha\mu}D_{\mu,\alpha\beta}(\partial^\nu A_{\beta\nu})=0, \qquad (5.2.54)$$

which cannot be expressed as an addition to the action functional. This gauge-fixing condition provides the damping terms for both the transverse mode and the longitudinal mode of the Parisi–Wu equation for non-Abelian gauge field as long as

$$\xi>0.$$

To see this, we begin with the Parisi–Wu equation for the "free" non-Abelian gauge field

$$\frac{\partial}{\partial\tau}A_{\alpha\mu}(x;\tau)=(\eta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)A_{\alpha\nu}(x;\tau)+\eta_{\alpha\mu}(x;\tau), \qquad (5.2.55)$$

$$\langle \eta_{\alpha\mu}(x;\tau) \rangle = 0,$$

$$\langle \eta_{\alpha\mu}(x;\tau)\eta_{\beta\nu}(x';\tau') \rangle = 2\delta_{\alpha\beta}\delta_{\mu\nu}\delta^4(x-x')\delta(\tau-\tau'). \tag{5.2.56}$$

We split (5.2.55) into a transverse mode and a longitudinal mode:

$$\frac{\partial}{\partial\tau} A^{\mathrm{T}}_{\alpha\mu}(x;\tau) = \partial^2 A^{\mathrm{T}}_{\alpha\mu}(x;\tau) + \eta^{\mathrm{T}}_{\alpha\mu}(x;\tau), \tag{5.2.57}$$

$$\frac{\partial}{\partial\tau} A^{\mathrm{L}}_{\alpha\mu}(x;\tau) = \eta^{\mathrm{L}}_{\alpha\mu}(x;\tau). \tag{5.2.58}$$

The absence of the damping term in the longitudinal equation, (5.2.58), implies that the initial distribution of the longitudinal mode persists in the gauge field "free" Green's function. We know that the width of the initial distribution of the longitudinal mode is a covariant gauge parameter. The divergent term proportional to $\tau$ emerges in the gauge field "free" Green's function. The longitudinal mode equation also produces the Faddeev–Popov ghost contribution.

If we introduce

$$\frac{1}{2\xi}(\partial^\mu A_{\alpha\mu}(x;\tau))^2 \tag{5.2.59}$$

into the Lagrangian density following the standard procedure, the longitudinal equation gets the additional term

$$\frac{1}{\xi}\partial_\mu(\partial^\nu A_{\alpha\nu}) \tag{5.2.60}$$

as the damping term. The additional term, (5.2.60), can be considered to come from the holonomic constraint

$$\delta \int \mathrm{d}^4x \frac{1}{2}(\partial^\mu A_{\alpha\mu}(x))^2 = -\int \mathrm{d}^4x \delta A_{\alpha\mu}(x)\partial^\mu(\partial^\nu A_{\alpha\nu}(x)) = 0, \tag{5.2.61}$$

in the classical field equation. Since we already know that (5.2.59) breaks gauge invariance and unitarity, we shall replace the holonomic constraint, (5.2.61), by the covariant nonholonomic constraint

$$\int \mathrm{d}^4x \delta A_{\alpha\mu}(x) D^\mu_{\alpha\beta}(\partial^\nu A_{\beta\nu}(x)) = 0, \tag{5.2.62}$$

with

$$D_{\mu,\alpha\beta} = \partial_\mu\delta_{\alpha\beta} + C_{\alpha\beta\gamma}A_{\gamma\mu}(x).$$

We call (5.2.62) the covariant gauge-fixing condition. With this covariant gauge-fixing condition, we have the classical field equation

$$\frac{\delta I_{\mathrm{E}}[A_{\beta\nu}]}{\delta A_{\alpha\mu}(x)} - \frac{1}{\xi} D_{\mu,\alpha\beta}(\partial^{\nu} A_{\beta\nu}(x)) = 0, \tag{5.2.63}$$

where $-\xi^{-1}$ is a Lagrange multiplier. Since

$$K^{\mu}_{\alpha} \equiv \frac{\delta I_{\mathrm{E}}[A_{\beta\nu}]}{\delta A_{\alpha\mu}(x)}$$

is orthogonal to

$$L_{\alpha\mu} \equiv D_{\mu,\alpha\beta}(\partial^{\nu} A_{\beta\nu}(x)),$$

i.e.,

$$(L_{\alpha\mu}, K^{\mu}_{\alpha}) = 0,$$

we obtain from the classical field equation, (5.2.63),

$$\frac{1}{\xi}\int \mathrm{d}^4 x |D_{\mu,\alpha\beta}(\partial^{\nu} A_{\beta\nu}(x))|^2 = 0,$$

which implies

$$D_{\mu,\alpha\beta}(\partial^{\nu} A_{\beta\nu}(x)) = 0 \qquad \text{almost everywhere in } x \text{ space.} \tag{5.2.64}$$

On the basis of the classical field equation, (5.2.63), we move on to stochastic quantization.

We have the Parisi–Wu equation as

$$\frac{\partial}{\partial\tau} A_{\alpha\mu}(x;\tau) = \left\{ \frac{\delta I_{\mathrm{E}}[A_{\beta\nu};\tau]}{\delta A^{\mu}_{\alpha}(x;\tau)} + \frac{1}{\xi} D_{\mu,\alpha\beta}(\partial^{\nu} A_{\beta\nu}(x;\tau)) \right\} + \eta_{\alpha\mu}(x;\tau). \tag{5.2.65}$$

We now derive the gauge field "free" Green's function. We keep only the relevant terms in (5.2.65) in deriving the gauge field "free" Green's function. Namely, we keep the terms linear in $A^{\nu}_{\alpha}(x;\tau)$ on the right-hand side of (5.2.65).

$$\frac{\partial}{\partial\tau} A_{\alpha\mu}(x;\tau) = (\delta_{\mu\nu}\partial^2 - \partial_{\mu}\partial_{\nu}) A^{\nu}_{\alpha}(x;\tau) + \frac{1}{\xi}\partial_{\mu}\partial_{\nu} A^{\nu}_{\alpha}(x;\tau) + \eta_{\alpha\mu}(x;\tau). \tag{5.2.66C}$$

In terms of the Fourier component with respect to $x$, we have

$$\begin{aligned}\frac{\partial}{\partial\tau}\tilde{A}_{\alpha\mu}(k;\tau) &= -k^2\left(\delta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}\right)\tilde{A}^{\nu}_{\alpha}(k;\tau) - \frac{k^2}{\xi}\frac{k_{\mu}k_{\nu}}{k^2}\tilde{A}^{\nu}_{\alpha}(k;\tau) + \tilde{\eta}_{\alpha\mu}(k;\tau) \\ &= -k^2 \tilde{A}^{(\mathrm{T})}_{\alpha\mu}(k;\tau) - \frac{k^2}{\xi}\tilde{A}^{(\mathrm{L})}_{\alpha\mu}(k;\tau) + \tilde{\eta}_{\alpha\mu}(k;\tau).\end{aligned} \tag{5.2.67}$$

Splitting (5.2.67) into a transverse mode and a longitudinal mode, we have

$$\frac{\partial}{\partial\tau}\tilde{A}^{(\mathrm{T})}_{\alpha\mu}(k;\tau) = -k^2\tilde{A}^{(\mathrm{T})}_{\alpha\mu}(k;\tau) + \tilde{\eta}^{(\mathrm{T})}_{\alpha\mu}(k;\tau), \tag{5.2.68T}$$

$$\frac{\partial}{\partial\tau}\tilde{A}^{(\mathrm{L})}_{\alpha\mu}(k;\tau) = -\frac{k^2}{\xi}\tilde{A}^{(\mathrm{L})}_{\alpha\mu}(k;\tau) + \tilde{\eta}^{(\mathrm{L})}_{\alpha\mu}(k;\tau). \tag{5.2.68L}$$

We note that the longitudinal mode of the gauge field has a damping term as long as

$$\xi > 0.$$

We solve (5.2.68T) and (5.2.68L) to obtain

$$\tilde{A}^{(\mathrm{T})}_{\alpha\mu}(k;\tau) = \exp[-k^2\tau]\int_0^\tau \mathrm{d}\tau' \exp[k^2\tau']\tilde{\eta}^{(\mathrm{T})}_{\alpha\mu}(k;\tau'), \tag{5.2.69T}$$

$$\tilde{A}^{(\mathrm{L})}_{\alpha\mu}(k;\tau) = \exp\left[-\frac{k^2}{\xi}\tau\right]\int_0^\tau \mathrm{d}\tau' \exp\left[\frac{k^2}{\xi}\tau'\right]\tilde{\eta}^{(\mathrm{L})}_{\alpha\mu}(k;\tau'), \tag{5.2.69L}$$

where the initial distribution terms are dropped from the beginning in the above solutions, anticipating the future limit $\tau \to \infty$. We calculate the correlation function:

$$\begin{aligned}
&\langle \tilde{A}_{\alpha\mu}(k;\tau)\tilde{A}_{\beta\nu}(k';\tau')\rangle \\
&= \delta_{\alpha\beta}\delta^4(k+k')\Bigg\{\frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)(\exp[k^2(\tau-\tau')] - \exp[-k^2(\tau+\tau')]) \\
&\quad + \xi\frac{k_\mu k_\nu}{(k^2)^2}\left(\exp\left[\frac{k^2}{\xi}(\tau-\tau')\right] - \exp\left[-\frac{k^2}{\xi}(\tau+\tau')\right]\right)\Bigg\}.
\end{aligned} \tag{5.2.70}$$

We have the gauge field "free" Green's function as

$$\begin{aligned}
&\lim_{\tau=\tau'\to\infty}\langle \tilde{A}_{\alpha\mu}(k;\tau)\tilde{A}_{\beta\nu}(k';\tau')\rangle \\
&\quad = \delta_{\alpha\beta}\delta^4(k+k')\frac{1}{k^2}\left\{\delta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right\},
\end{aligned} \tag{5.2.71}$$

which is free from the initial distribution term of the longitudinal mode and the divergent term proportional to $\tau$. The Lagrange multiplier

$$\xi > 0$$

of the covariant nonholonomic gauge-fixing condition is the covariant gauge parameter.

The second term of the covariant nonholonomic gauge fixing condition, (5.2.64), provides the Faddeev–Popov ghost contribution. The transverse mode of the Parisi–Wu equation, (5.2.65), provides the ordinary gluon contribution.

We discuss the Fokker–Planck equation for a non-Abelian gauge field with the constraint, (5.2.64). Following the procedure employed in Sect. 5.2.1, we can derive the modified Fokker–Planck equation for a non-Abelian gauge field:

$$\frac{\partial}{\partial \tau}\psi[A;\tau] = (\mathcal{F}_0 + \mathcal{F}_\xi)\psi[A;\tau], \tag{5.2.72}$$

where the Fokker–Planck operators, $\mathcal{F}_0$ and $\mathcal{F}_\xi$, are respectively given by

$$\mathcal{F}_0 = \int \mathrm{d}^4x \frac{\delta}{\delta A_{\alpha\mu}(x)}\left[\frac{\delta}{\delta A_{\alpha\mu}(x)} - \frac{\delta I_{\mathrm{E}}[A_{\beta\nu}]}{\delta A_{\alpha\mu}(x)}\right], \tag{5.2.73}$$

$$\begin{aligned}\mathcal{F}_\xi &= \int \mathrm{d}^4x \frac{\delta}{\delta A_{\alpha\mu}(x)}\left[\frac{1}{\xi}D_{\mu,\alpha\beta}(\partial^\nu A_{\beta\nu}(x))\right] \\ &= \frac{1}{\xi}\left[\int \mathrm{d}^4x(\partial^\mu A_{\alpha\mu}(x))G_\alpha(x)\right]^\dagger,\end{aligned} \tag{5.2.74}$$

with $G_\alpha(x)$ defined by

$$G_\alpha(x) \equiv -D_{\mu,\alpha\beta}\left(\frac{\delta}{\delta A_{\beta\mu}(x)}\right), \tag{5.2.75}$$

being the generator of the gauge transformation. The $\mathcal{F}_\xi$ term in the Fokker–Planck equation, (5.2.72), originates from the covariant nonholonomic constraint, (5.2.64). The constraint leaves the gauge invariant quantity $\Phi$ unchanged, i.e.,

$$G_\alpha \Phi = 0. \tag{5.2.76}$$

In this sense, the covariant nonholonomic constraint, (5.2.64), never breaks the gauge invariance.

The modified Fokker–Planck equation, (5.2.72), for a non-Abelian gauge field has the stationary solution,

$$\psi^{(\mathrm{eq})}[A_{\alpha\mu}] = C \cdot \delta(D_{\mu,\alpha\beta}(\partial^\nu A_{\beta\nu}))\exp\left[\hat{\beta} I_{\mathrm{E}}[A_{\alpha\mu}]\right], \tag{5.2.77}$$

with the normalization constant $C$ given by

$$C = \left\{\int \mathcal{D}[A_{\alpha\mu}]\delta(D_{\mu,\alpha\beta}(\partial^\nu A_{\beta\nu}))\exp\left[\hat{\beta} I_{\mathrm{E}}[A_{\alpha\mu}]\right]\right\}^{-1}.$$

The covariant nonholonomic gauge-fixing condition, (5.2.64), is enforced on the stationary solution by the presence of the delta function,

$$\delta(D_{\mu,\alpha\beta}(\partial^{\nu}A_{\beta\nu})),$$

in (5.2.77). We have formulated non-Abelian gauge field theory as a stationary state of a stochastic process.

## 5.3 Bibliography

### Chapter 5. Section 5.1

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(**T-61**) Parzan, E.; "Stochastic Processes", Holden-Day, 1962, San Francisco.
(**T-62**) Karlin, S. and Taylor, H.M.; "A First Course in Stochastic Processes", 1975, Academic Press, New York.
(**T-63**) Karlin, S. and Taylor, H.M.; "A Second Course in Stochastic Processes", 1981, Academic Press, New York.

The starting point of the *stochastic quantization* is the article by G. Parisi and Y. Wu.

(**R5-1**) Parisi, G. and Wu, Y.; Sci. Sin. **24**, (1981), 483.
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### Chapter 5. Section 5.2

Stochastic quantization of a non-Abelian gauge field is discussed in the following article, with the extensive perturbative calculation.

(**R5-2**) Namiki, M., et al.; Prog. Theor. Phys. **69**, (1983), 1580.

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(**R5-3**) Fukai, T., et al.; Prog. Theor. Phys. **69**, (1983), 1600.

The derivation of the Fokker–Planck equation from the Parisi–Wu equation and the stationary solution of the Fokker–Planck equation are discussed in the following articles.

(**R5-4**) Namiki, M. and Yamanaka, Y.; Prog. Theor. Phys. **69**, (1983), 1764.
(**R5-5**) Zwanziger, D.; Nucl. Phys. **B192**, (1981), 259.
(**R5-6**) Baulieu, L. and Zwanziger, D.; Nucl. Phys. **B193**, (1981), 163.

Covariant non-holonomic gauge fixing condition in the stochastic quantization of non-Abelian-gauge field theory is discussed in the following article.

(**R5-7**) Nakagoshi, H., et al.; Prog. Theor. Phys. **70**, (1983), 326.

Stochastic quantization of a dynamical system with holonomic constraints is discussed in the following article, together with the results of a numerical simulation.

(**R5-8**) Namiki, M., et al.; Prog. Theor. Phys. **72**, (1984), 350.

Stochastic regularization and renormalization are discussed in the following article, with the fundamental modifications of the Parisi–Wu equation.

(**R5-9**) Namiki, M. and Yamanaka, Y.; Hadronic J. **7**, (1984), 594.

Namiki and Yamanaka use the *Ornstein–Uhlenbeck process* as the driving stochastic process, which can be derived from the Gaussian white noise. However, the original Parisi–Wu equation gets modified to the second order partial differential equation with respect to the evolution parameter.

Stochastic quantization and regularization for quartic self-interacting scalar field theory and non-Abelian gauge field theory in the *6-dimensional space-time* are carried out in the following articles.

(**R5-10**) McClain, B., Niemi, A. and Taylor, C.; Ann. Phys. **140**, (1982), 232.
(**R5-11**) Niemi, A. and Wijewardhana, L.C.R.; Ann. Phys. **140**, (1982), 247.

# A.1 Gaussian Integration

By the Poisson method, we have the Gaussian integral formula.

$$G(a) = \int_{-\infty}^{+\infty} \mathrm{d}\xi \exp\left[-\frac{a}{2}\xi^2\right] = \sqrt{\frac{2\pi}{a}}, \qquad \mathrm{Re}\, a > 0. \tag{A.1.1}$$

We extend (A1.1) to $f$ degrees of freedom. We let

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_f)^{\mathrm{T}}, \qquad \int \mathrm{d}\boldsymbol{\xi} = \int_{-\infty}^{+\infty} \prod_{r=1}^{f} \mathrm{d}\xi_r, \tag{A.1.2a}$$

$$D = f \times f \text{ real symmetric positive definite matrix.} \tag{A.1.2b}$$

We define $G(D)$ by

$$G(D) \equiv \int \mathrm{d}\boldsymbol{\xi} \exp\left[-\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}} D \boldsymbol{\xi}\right] \equiv \int_{-\infty}^{+\infty} \prod_{r=1}^{f} \mathrm{d}\xi_r \exp\left[-\frac{1}{2}\sum_{r,s=1}^{f} D_{r,s}\xi_r\xi_s\right]. \tag{A.1.3}$$

We let the orthogonal matrix $R$ diagonalize $D$. We then have

$$D = R^{\mathrm{T}}\tilde{D}R, \qquad R^{\mathrm{T}}R = RR^{\mathrm{T}} = 1, \qquad |\det R| = 1, \tag{A.1.4a}$$

$$\tilde{D} = \mathrm{diag}(d_1, \ldots, d_f). \tag{A.1.4b}$$

We evaluate $G(D)$ as

$$\begin{aligned} G(D) &= \int \mathrm{d}(R\boldsymbol{\xi}) \exp\left[-\frac{1}{2}(R\boldsymbol{\xi})^{\mathrm{T}}\tilde{D}(R\boldsymbol{\xi})\right] \\ &= \int \mathrm{d}\boldsymbol{\zeta} \exp\left[-\frac{1}{2}\boldsymbol{\zeta}^{\mathrm{T}}\tilde{D}\boldsymbol{\zeta}\right] \\ &\qquad\qquad (\boldsymbol{\zeta} = R\boldsymbol{\xi}, \qquad |\det R| = 1,) \end{aligned}$$

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STMP 165, 245–273 (2000)

$$
\begin{aligned}
&= \prod_{r=1}^{f} G(d_r) \\
&= \prod_{r=1}^{f} (2\pi)^{1/2} d_r{}^{-1/2} \\
&= (2\pi)^{f/2} \left( \prod_{r=1}^{f} d_r \right)^{-1/2} \\
&= (2\pi)^{f/2} (\det \tilde{D})^{-1/2} \\
&= (2\pi)^{f/2} (\det D)^{-1/2}. \qquad \text{(A.1.5)}
\end{aligned}
$$

We define the ill-defined quasi-Gaussian integral $G(\mathrm{i}D)$ formally by

$$
\begin{aligned}
G(\mathrm{i}D) &\equiv \lim_{\varepsilon \to 0^+} G((\mathrm{i} + \varepsilon) D) \\
&= \lim_{\varepsilon \to 0^+} \prod_{r=1}^{f} G((i + \varepsilon) d_r) \\
&= \lim_{\varepsilon \to 0^+} \left( \frac{2\pi}{\mathrm{i} + \varepsilon} \right)^{f/2} (\det D)^{-1/2} \\
&= (-2\pi \mathrm{i})^{f/2} (\det D)^{-1/2}. \qquad \text{(A.1.6)}
\end{aligned}
$$

Under (A.1.2a), we complete the square of the following expression

$$
\begin{aligned}
&\frac{1}{2} \sum_{r,s=1}^{f} D_{r,s} \xi_r \xi_s - \sum_{r=1}^{f} C_r \xi_r \\
&= \frac{1}{2} \sum_{r,s=1}^{f} D_{r,s} (\xi_r - \eta_r)(\xi_s - \eta_s) - \frac{1}{2} \sum_{r,s=1}^{f} D_{r,s} \eta_r \eta_s \qquad \text{(A.1.7)}
\end{aligned}
$$

where

$$
\eta_r \equiv (D^{-1})_{r,s} C_s, \qquad \text{(A.1.8)}
$$

and

$$
-\frac{1}{2} \sum_{r,s=1}^{f} D_{r,s} \eta_r \eta_s = -\frac{1}{2} \sum_{r,s=1}^{f} (D^{-1})_{r,s} C_r C_s. \qquad \text{(A.1.9)}
$$

Using (A.1.6), we obtain the formula

$$\int \mathrm{d}\xi \exp\left[-\frac{\mathrm{i}}{2}\sum_{r,s=1}^{f} D_{r,s}\xi_r\xi_s - \mathrm{i}\sum_{r=1}^{f} C_r\xi_r\right]$$
$$= (-2\pi\mathrm{i})^{f/2}(\det D)^{-1/2}\exp\left[-\frac{\mathrm{i}}{2}\sum_{r,s=1}^{f}(D^{-1})_{r,s}C_rC_s\right]. \qquad \text{(A.1.10)}$$

An easy way to remember (A.1.10) is the following:

$$\eta_r = \xi_r \equiv \text{stationary point of (A.1.7)}, \qquad \text{(A.1.11a)}$$

$$-\frac{1}{2}\sum_{r,s=1}^{f}(D^{-1})_{r,s}C_rC_s \equiv \text{stationary value of (A.1.7)}. \qquad \text{(A.1.11b)}$$

Next, we state Wick's theorem for finite degrees of freedom:

$$\int \mathrm{d}\xi\xi_{a_1}\cdots\xi_{a_{2N}}\exp\left[-\frac{\mathrm{i}}{2}\sum_{r,s=1}^{f} D_{r,s}\xi_r\xi_s\right]$$
$$= (-2\pi\mathrm{i})^{f/2}(\det D)^{-1/2}\sum_{\substack{\text{All possible}\\ \text{pairing } (a,a')\\ \text{of } (a_1,\ldots,a_{2N})}}\ \prod_{\substack{\text{particular}\\ \text{pairing}}}^{(a,a')}\left(\frac{1}{\mathrm{i}}D^{-1}\right)_{a,a'}. \qquad \text{(A.1.12)}$$

This formula can be proven as

The left-hand side of (A.1.12)

$$= \mathrm{i}\frac{\partial}{\partial C_{a_1}}\cdots\mathrm{i}\frac{\partial}{\partial C_{a_{2N}}}\{\text{the left-hand side of (A.1.10)}\}\,|_{C\equiv 0}$$

= the right-hand side of (A.1.12).

In the case when $\xi$ is a complex number, and $D$ is a Hermitian positive definite matrix, we have

$$\int \mathrm{d}\boldsymbol{\xi}^*\mathrm{d}\boldsymbol{\xi}\exp[-\mathrm{i}\boldsymbol{\xi}^\dagger D\boldsymbol{\xi}] = \int\prod_{r=1}^{f}\mathrm{d}(\mathrm{Re}\,\xi_r)\mathrm{d}(\mathrm{Im}\,\xi_r)\exp\left[-\mathrm{i}\sum_{r,s=1}^{f}\xi_r^* D_{r,s}\xi_s\right]$$
$$= (-2\pi\mathrm{i})^f(\det D)^{-1}. \qquad \text{(A.1.13)}$$

Lastly, we consider a singular Gaussian integral. When

$$\det D = 0, \tag{A.1.14}$$

i.e., when the matrix $D$ has the eigenvalue 0, $G(D)$ given by (A.1.5) is divergent. We shall consider the procedure of giving a mathematical meaning to $G(D)$ under (A.1.14). We assume that the $f \times f$ matrix $D$ has $p$ zero eigenvalues:

$$d_{f-p+1} = \cdots = d_f = 0. \tag{A.1.15}$$

As a restricted $G(D)$, we consider $G_{\text{rest}}(D)$ defined by

$$G_{\text{rest}}(D) \equiv \int_{-\infty}^{+\infty} \prod_{r=1}^{f-p} \mathrm{d}\eta_r \exp\left[-\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}(\boldsymbol{\eta}) D \boldsymbol{\xi}(\boldsymbol{\eta})\right]. \tag{A.1.16}$$

$\{\eta_r\}_{r=1}^{f-p}$ is an orthogonal coordinate system corresponding to the nonzero eigenvalues $\{d_r\}_{r=1}^{f-p}$ of $D$. The right-hand side of (A.1.16) depends on the choice of coordinate system $\{\eta_r\}_{r=1}^{f-p}$. We introduce the dummy variables $\{\eta_r\}_{r=f-p+1}^{f}$, and rewrite (A.1.16) as

$$G_{\text{rest}}(D) = \int_{-\infty}^{+\infty} \prod_{r=1}^{f} \mathrm{d}\eta_r \prod_{r=f-p+1}^{f} \delta(\eta_r) \exp\left[-\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}(\boldsymbol{\eta}) D \boldsymbol{\xi}(\boldsymbol{\eta})\right]. \tag{A.1.17}$$

We now perform a change of variables from $\{\eta_r\}_{r=1}^{f}$ to $\{\xi_r\}_{r=1}^{f}$. With the use of the formula for the change of variables

$$\prod_{r=1}^{f} \mathrm{d}\eta_r = \prod_{r=1}^{f} \mathrm{d}\xi_r \det\left(\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}}\right), \tag{A.1.18}$$

we cast (A.1.17) into the form

$$G_{\text{rest}}(D) = \int_{-\infty}^{+\infty} \prod_{r=1}^{f} \mathrm{d}\xi_r \det\left(\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}}\right) \prod_{r=f-p+1}^{f} \delta(\eta_r) \exp\left[-\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}} D \boldsymbol{\xi}\right]. \tag{A.1.19}$$

We observe that this integral is well defined. The dummy variables, $\{\eta_r\}_{r=f-p+1}^{f}$, of the $\delta$ functions are arbitrary functions of $\{\xi_r\}_{r=1}^{f}$. The extra factor in the integral measure of (A.1.19)

$$\det\left(\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}}\right) \prod_{r=f-p+1}^{f} \delta(\eta_r) \tag{A.1.20}$$

lowers the integral from an $f$-dimensional integral down to a $(f-p)$-dimensional integral. $G_{\mathrm{rest}}(D)$ does not depend on the choice of $\{\eta_r(\boldsymbol{\xi})\}_{r=1}^{f-p}$. We choose $\{\eta_r(\boldsymbol{\xi})\}_{r=1}^{f-p}$ such that

$$\det\left(\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{\xi}}\right) \neq 0. \tag{A.1.21}$$

The quasi-Gaussian integral $G_{\mathrm{rest}}(\mathrm{i}D)$ is also defined formally like (A.1.6) from (A.1.19).

Formula (A.1.19) is the finite-dimensional version of the Faddeev–Popov formula used in Chap. 3. In the language of path integral quantization of non-Abelian gauge field theory,

$$\prod_{r=f-p+1}^{f} \delta(\eta_r) \tag{A.1.22}$$

fixes the "gauge" and

$$\det\left(\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{\xi}}\right) \tag{A.1.23}$$

corresponds to the Faddeev–Popov determinant.

## A.2 Fermion Number Integration

We consider the complex Grassman variables, $\eta_k$, $k = 1, \dots, f$.

$$\boldsymbol{\eta} = (\eta_1, \dots, \eta_f)^{\mathrm{T}}. \tag{A.2.1}$$

We have the following properties for the complex Grassman variables $\eta_k$:

$$\{\eta_k, \eta_l\} = \{\eta_k^*, \eta_l^*\} = \{\eta_k, \eta_l^*\} = 0, \qquad \eta_k^2 = \eta_l^{*2} = 0, \qquad [\eta_k, q_l] = 0. \tag{A.2.2a}$$

$$\left\{\frac{\partial}{\partial\eta_k}, \eta_l\right\} = \left\{\frac{\partial}{\partial\eta_k^*}, \eta_l^*\right\} = \delta_{kl}, \qquad \left\{\frac{\partial}{\partial\eta_k}, \eta_l^*\right\} = \left\{\frac{\partial}{\partial\eta_k^*}, \eta_l\right\} = 0. \tag{A.2.2b}$$

$$\left\{\frac{\partial}{\partial\eta_k}, \frac{\partial}{\partial\eta_l}\right\} = \left\{\frac{\partial}{\partial\eta_k^*}, \frac{\partial}{\partial\eta_l^*}\right\} = \left\{\frac{\partial}{\partial\eta_k^*}, \frac{\partial}{\partial\eta_l}\right\} = 0. \tag{A.2.2c}$$

$$\int \mathrm{d}\eta_k 1 = \int \mathrm{d}\eta_l^* 1 = 0, \qquad \int \mathrm{d}\eta_k \eta_k = \int \mathrm{d}\eta_l^* \eta_l^* = 1. \tag{A.2.3}$$

We consider the function $g(\eta, \eta^*)$ defined by

$$g(\eta, \eta^*) = \sum_{k,l} \eta_{a_1}^* \cdots \eta_{a_k}^* \eta_{b_1} \cdots \eta_{b_l} g_{a_1,\dots,a_k;b_1,\cdots,b_l}^{(k,l)} \tag{A.2.4}$$

This power series expansion gets terminated at

$$k = l = f.$$

The expansion coefficients, $g_{a_1,\dots,a_k;b_1,\dots,b_l}^{(k,l)}$, are completely antisymmetric with respect to $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_l\}$, respectively. Using (A.2.2a), (A.2.2b), (A.2.2c) and (A.2.3), we obtain the integral formula:

$$\int \prod_{k=1}^{f} \mathrm{d}\eta_k \prod_{l=1}^{f} \mathrm{d}\eta_l^* g(\eta, \eta^*) = \sum_{\pi a, \pi b} g_{\pi\{a_1,\dots,a_f;b_1,\dots,b_f\}}^{(f,f)} \delta_{\pi a} \delta_{\pi b} \tag{A.2.5}$$

$$= (f!)^2 g_{1,\dots,f;1,\dots,f}^{(f,f)}, \tag{A.2.6}$$

where $\delta_{\pi a}$ and $\delta_{\pi b}$ are the signatures of the permutations, $\pi a$ and $\pi b$, respectively.

We let $D$ to be a Hermitian positive definite matrix, and let $g(\eta, \eta^*)$ be

$$g(\eta, \eta^*) \equiv \exp\left[\mathrm{i}\sum_{n,m=1}^{f} D_{nm}\eta_n^*\eta_m\right]\begin{pmatrix}1\\ \eta_c^*\eta_d\end{pmatrix}. \tag{A.2.7}$$

With the use of (A.2.6) and the definitions of $\det D$ and $D^{-1}$, we obtain

$$\int \prod_{k=1}^{f} \mathrm{d}\eta_k \prod_{l=1}^{f} \mathrm{d}\eta_l^* \exp\left[\mathrm{i}\sum_{n,m=1}^{f} D_{nm}\eta_n^*\eta_m\right]\begin{pmatrix}1\\ \eta_c^*\eta_d\end{pmatrix}$$

$$= (\mathrm{i})^f(-1)^{f(f-1)/2}\cdot\det D\cdot\begin{pmatrix}1\\ \left(\frac{1}{\mathrm{i}}D^{-1}\right)_{cd}\end{pmatrix}. \tag{A.2.8}$$

As an application of this formula, we have the Grassman algebra version of (A.1.10):

$$\int \prod_{k=1}^{f} \mathrm{d}\eta_k \prod_{l=1}^{f} \mathrm{d}\eta_l^* \exp\left[\mathrm{i}\sum_{n,m=1}^{f} D_{nm}\eta_n^*\eta_m + i\sum_{n=1}^{f}(\eta_n^*\zeta_n + \zeta_n^*\eta_n)\right] \tag{A.2.9}$$

$$= (\mathrm{i})^f(-1)^{f(f-1)/2}\cdot\det D\cdot\exp\left[-\mathrm{i}\sum_{n,m=1}^{f}(D^{-1})_{nm}\zeta_n^*\zeta_m\right].$$

The way to remember (A.2.9) is the same as (A.1.10).

Lastly, we record here the Grassman algebra version of Wick's theorem in the finite dimension:

$$\int \prod_{k=1}^{f} \mathrm{d}\eta_k \prod_{l=1}^{f} \mathrm{d}\eta_l^* (\eta_{c_N}^* \cdots \eta_{c_1}^*\eta_{d_1}\cdots\eta_{d_N})\exp\left[\mathrm{i}\sum_{n,m=1}^{f} D_{nm}\eta_n^*\eta_m\right]$$

$$= (\mathrm{i})^f(-1)^{f(f-1)/2}\cdot\det D\cdot \sum_{\substack{\text{all possible}\\ \text{pairing of}\\ (c_1,\ldots,c_N;d_1,\ldots,d_N)}} \prod_{\substack{\text{particular}\\ \text{pairing}}}^{(c,d)} \left(\frac{1}{\mathrm{i}}D^{-1}\right)_{c,d}\delta_{\pi(c,d)}. \tag{A.2.10}$$

It is worthwhile to keep in mind that we have

$\det D$

instead of

$$(\det D)^{-1/2}$$

in (A.2.8), (A.2.9) and (A.2.10), which is due to the anticommutativity of the Grassman number.

# A.3 Functional Integration

We regard field theory as a mechanical system with infinite degrees of freedom. We can consider the functional integral in quantum theory in the following two senses.

1. *Limit of discretized space-time:*

(1) We discretize Minkowski space-time into $n_0$ cells.
(2) We let the averaged values of the field quantity $\alpha(x)$ on each cell be represented by

$$\alpha_i, \qquad i = 1, \ldots, n_0,$$

and regard $\{\alpha_i\}_{i=1}^{n_0}$ as independent dynamical variables.
(3) We perform the ordinary integral

$$\int_{-\infty}^{+\infty} \mathrm{d}\alpha_i$$

on each cell $i$.
(4) We take the limit $n_0 \to \infty$ (cell volume$\to 0$).

2. *Friedrichs integral:*

(1) We expand the field quantity $\alpha(x)$ into the normal modes $\{\alpha_\lambda\}_{\lambda=0}^{\infty}$ of the complete orthonormal system $\{u_\lambda(x)\}_{\lambda=0}^{\infty}$ which spans the Hilbert space.
(2) We let the expansion coefficients $\{\alpha_\lambda\}_{\lambda=0}^{\infty}$ be the independent integration variables.
(3) We truncate $\{\alpha_\lambda\}_{\lambda=0}^{\infty}$ at $\lambda = N_0$ and perform the ordinary $N_0$-tuple integration.
(4) We take the limit $N_0 \to \infty$.

We employed the procedure 1 when we extended quantum mechanics with finite degrees of freedom to quantum field theory with infinite degrees of freedom in Sect. 2.1 of Chap. 2. In this appendix, we discuss procedure 2.

We consider the functional integral

$$\int \mathcal{D}[\alpha] F[\alpha] \tag{A.3.1}$$

of the functional $F[\alpha]$ of the square integrable real function $\alpha(x)$. We expand the real function $\alpha(x)$ in terms of the complete orthonormal system $\{u_\lambda(x)\}_{\lambda=0}^{\infty}$.

$$\alpha(x) = \sum_{\lambda=0}^{\infty} \alpha_\lambda u_\lambda(x), \tag{A.3.2a}$$

$$\alpha_\lambda = \int_{-\infty}^{+\infty} \mathrm{d}x \alpha(x) u_\lambda(x). \tag{A.3.2b}$$

We regard $\{\alpha_\lambda\}_{\lambda=0}^{\infty}$ as independent integration variables. When we truncate the expansion coefficients $\{\alpha_\lambda\}_{\lambda=0}^{\infty}$ at $\lambda = N_0$, the functional $F[\alpha]$ becomes an ordinary function $F(\alpha_1, \ldots, \alpha_{N_0})$ of the $N_0$ variables and (A.3.1) is defined by

$$\int \mathcal{D}[\alpha] F[\alpha] = \lim_{N_0 \to \infty} \cdots \int \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{N_0} F(\alpha_1, \ldots, \alpha_{N_0}). \tag{A.3.3}$$

**Trivial example:** For the Gaussian functional

$$\exp\left[-\frac{1}{2}\alpha^2\right] \equiv \exp\left[-\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x \alpha^2(x)\right], \tag{A.3.4}$$

we expand $\alpha(x)$ in terms of $\{u_\lambda(x)\}_{\lambda=0}^{\infty}$, and obtain

$$\int_{-\infty}^{+\infty} \mathrm{d}x \alpha^2(x) = \sum_{\lambda=0}^{\infty} \alpha_\lambda^2. \tag{A.3.5}$$

The Gaussian functional integral

$$\int \mathcal{D}[\alpha] \exp\left[-\tfrac{1}{2}\alpha^2\right] \tag{A.3.6}$$

is evaluated by defining the functional integral measure $\mathcal{D}[\alpha]$ as

$$\mathcal{D}[\alpha] \equiv \lim_{N_0 \to \infty} \prod_{\lambda=0}^{N_0} \frac{\mathrm{d}\alpha_\lambda}{\sqrt{2\pi}}. \tag{A.3.7}$$

We then have

$$\begin{aligned} \int D[\alpha] \exp\left[-\tfrac{1}{2}\alpha^2\right] &= \lim_{N_0 \to \infty} \int_{-\infty}^{+\infty} \prod_{\lambda=0}^{N_0} \frac{\mathrm{d}\alpha_\lambda}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\sum_{\lambda=0}^{N_0} \alpha_\lambda^2\right] \\ &= \lim_{N_0 \to \infty} \prod_{\lambda=0}^{N_0} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\alpha_\lambda}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\alpha_\lambda^2\right] \\ &= 1. \end{aligned} \tag{A.3.8}$$

For the application of the functional integral method to quantum theory, we only need two more assumptions besides the result (A.3.8).

*Assumption 1.* The functional integral is "translationally invariant".

$$\int \mathcal{D}[\alpha]F[\alpha] = \int \mathcal{D}[\alpha]F\left[\alpha + \phi\right]. \tag{A.3.9}$$

*Assumption 2.* The functional integral depends on the functional integrand linearly so that the ordinary rules of calculus apply to functional integrals.

From (A.3.8) and assumption 1, we obtain

$$\int \mathcal{D}[\alpha] \exp\left[-\tfrac{1}{2}(\alpha + \phi)^2\right] = 1. \tag{A.3.10}$$

Expanding the exponent of (A.3.10), we obtain

$$\int \mathcal{D}[\alpha] \exp[-\phi\alpha - \tfrac{1}{2}\alpha^2] = \exp\left[\tfrac{1}{2}\phi^2\right].$$

Replacing $\phi$ by $-\mathrm{i}\phi$, we obtain

$$\int \mathcal{D}[\alpha] \exp[\mathrm{i}\phi\alpha - \tfrac{1}{2}\alpha^2] = \exp\left[-\tfrac{1}{2}\phi^2\right]. \tag{A.3.11}$$

From (A.3.11) and assumption 2, we obtain

$$\int \mathcal{D}[\alpha]F[\alpha] \exp[\mathrm{i}\phi\alpha - \tfrac{1}{2}\alpha^2] = F\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\phi}\right) \exp\left[-\tfrac{1}{2}\phi^2\right]. \tag{A.3.12}$$

Next, we observe that for real functions $\alpha(x)$ and $\beta(x)$,

$$\int \mathcal{D}[\alpha]\mathcal{D}[\beta] \exp\left[-\tfrac{1}{2}(\alpha^2 + \beta^2)\right] = 1. \tag{A.3.13}$$

We define the complex valued functions $\varphi(x)$ and $\varphi^*(x)$, and the corresponding functional integral measure $\mathcal{D}[\varphi]\mathcal{D}[\varphi^*]$ by

$$\varphi(x) \equiv \frac{1}{\sqrt{2}}(\alpha(x) + \mathrm{i}\beta(x)), \quad \varphi^*(x) \equiv \frac{1}{\sqrt{2}}(\alpha(x) - \mathrm{i}\beta(x)), \tag{A.3.14}$$

$$\frac{\mathrm{d}\varphi_\lambda}{\sqrt{\pi}}\frac{\mathrm{d}\varphi^*_\lambda}{\sqrt{\pi}} \equiv \frac{\mathrm{d}(\mathrm{Re}\,\varphi(x))}{\sqrt{\pi}}\frac{\mathrm{d}(\mathrm{Im}\,\varphi(x))}{\sqrt{\pi}} = \frac{\mathrm{d}\alpha_\lambda}{\sqrt{2\pi}}\frac{d\beta_\lambda}{\sqrt{2\pi}}, \tag{A.3.15}$$

$$\begin{aligned} \mathcal{D}[\varphi]\mathcal{D}[\varphi^*] &\equiv \lim_{N_0\to\infty}\prod_{\lambda=0}^{N_0}\frac{\mathrm{d}\varphi_\lambda}{\sqrt{\pi}}\frac{\mathrm{d}\varphi^*_\lambda}{\sqrt{\pi}} \\ &= \lim_{N_0\to\infty}\prod_{\lambda=0}^{N_0}\frac{\mathrm{d}\alpha_\lambda}{\sqrt{2\pi}}\frac{d\beta_\lambda}{\sqrt{2\pi}} \\ &= \mathcal{D}[\alpha]\mathcal{D}\left[\beta\right]. \end{aligned} \tag{A.3.16}$$

From (A.3.13), we have

$$\int \mathcal{D}[\varphi]\mathcal{D}[\varphi^*] \exp[-\varphi^*\varphi] = 1. \tag{A.3.17}$$

Here, we record the two-function-variables version of (A.3.11).

$$\int \mathcal{D}[\alpha]\mathcal{D}[\beta] \exp[\mathrm{i}(\gamma\alpha + \zeta\beta) - \tfrac{1}{2}(\alpha^2 + \beta^2)] = \exp\left[-\tfrac{1}{2}(\gamma^2 + \zeta^2)\right]. \tag{A.3.18}$$

We define the complex functions $\chi(x)$ and $\chi^*(x)$ of a real variable by

$$\chi(x) = \frac{1}{\sqrt{2}}(\gamma(x) + \mathrm{i}\zeta(x)), \quad \chi^*(x) = \frac{1}{\sqrt{2}}(\gamma(x) - \mathrm{i}\zeta(x)). \tag{A.3.19}$$

From (A.3.14), (A.3.18) and (A.3.19), we obtain

$$\int \mathcal{D}[\varphi]\mathcal{D}[\varphi^*] \exp[\mathrm{i}(\chi^*\varphi + \varphi^*\chi) - \varphi^*\varphi] = \exp\left[-\chi^*\chi\right]. \tag{A.3.20}$$

From (A.3.20), for the same reason as that applied to (A.3.12), we obtain

$$\int \mathcal{D}[\varphi]\mathcal{D}[\varphi^*] F[\varphi, \varphi^*] \exp[\mathrm{i}(\chi^*\varphi + \varphi^*\chi) - \varphi^*\varphi]$$
$$= F\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\chi^*}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta\chi}\right) \exp[-\chi^*\chi]. \tag{A.3.21}$$

**Functional Fourier Transform:** We define the functional Fourier transform $\tilde{F}[\omega]$ of the functional $F[\alpha]$ of the real function $\alpha(x)$ by

$$F[\alpha] \equiv \int \mathcal{D}[\omega]\tilde{F}[\omega] \exp\left[\mathrm{i}\omega\alpha\right], \tag{A.3.22}$$

and its inverse by

$$\tilde{F}[\omega] \equiv \int \mathcal{D}[\alpha]F[\alpha] \exp\left[-\mathrm{i}\omega\alpha\right]. \tag{A.3.23}$$

We define the $\delta$-functional $\delta[\alpha - \beta]$ by

$$\delta[\alpha - \beta] \equiv \int \mathcal{D}[\omega] \exp\left[\mathrm{i}\omega(\alpha - \beta)\right], \tag{A.3.24a}$$

with the property

$$\int \mathcal{D}[\alpha]F[\alpha]\delta[\alpha - \beta] = F\left[\beta\right], \tag{A.3.24b}$$

just like the ordinary $\delta$-function $\delta(x)$.

We define the functional Fourier transform $\tilde{F}[\chi^*, \chi]$ of the functional $F[\varphi, \varphi^*]$ of the complex functions $\varphi(x)$ and $\varphi^*(x)$ of the real variable by

$$F[\varphi, \varphi^*] \equiv \int \mathcal{D}[\chi]\mathcal{D}[\chi^*]\tilde{F}[\chi^*, \chi] \exp\left[\mathrm{i}(\chi^*\varphi + \varphi^*\chi)\right], \tag{A.3.25}$$

and its inverse by

$$\tilde{F}[\chi^*, \chi] \equiv \int \mathcal{D}[\varphi]\mathcal{D}[\varphi^*]F[\varphi, \varphi^*] \exp\left[-\mathrm{i}(\chi^*\varphi + \varphi^*\chi)\right]. \tag{A.3.26}$$

**Change of Function Variables:** We consider the change of function variables of the real function $\alpha(x)$ defined by

$$\alpha'(x) = \int_{-\infty}^{+\infty} \mathrm{d}y K(x, y)\alpha(y). \tag{A.3.27}$$

The kernel $K(x, y)$ is real and symmetric. We expand $\alpha'(x)$, $K(x, y)$ and $\alpha(x)$ in terms of the finite orthonormal system $\{u_\lambda(x)\}_{\lambda=0}^{N_0}$ as

$$\alpha'_\lambda = \sum_{\lambda'=0}^{N_0} k_{\lambda\lambda'}\alpha_{\lambda'}, \tag{A.3.28}$$

$$k_{\lambda\lambda'} = \int_{-\infty}^{+\infty} \mathrm{d}x\mathrm{d}x' u_\lambda(x)K(x, x')u_{\lambda'}(x') \tag{A.3.29a}$$

$$\equiv u_{\lambda,x}K_{x,x'}u_{\lambda',x'} = (uKu^{\mathrm{T}})_{\lambda,\lambda'} \tag{A.3.29b}$$

From (A.3.7), we take the limit $N_0 \to \infty$ in (A.3.28) and (A.3.29) and obtain

$$\mathcal{D}[K\alpha] = \det K \cdot \mathcal{D}\left[\alpha\right], \tag{A.3.30}$$

and

$$\det K_{x,x'} = \det k_{\lambda,\lambda'}. \tag{A.3.31}$$

Equation (A.3.31) follows from the fact that $\{u_\lambda(x)\}_{\lambda=0}^{\infty}$ forms a complete set of the orthonormal system, i.e., we have

$$u^{\mathrm{T}} = u^{-1}. \tag{A.3.32}$$

In the case of (A.3.14),

$$\varphi(x) = \frac{1}{\sqrt{2}}(\alpha(x) + \mathrm{i}\beta(x)), \quad \varphi^*(x) = \frac{1}{\sqrt{2}}(\alpha(x) - \mathrm{i}\beta(x)), \tag{A.3.14}$$

we perform a change of function variables to $\alpha'(x)$ and $\beta'(x)$.

$$\alpha'(x) = \int_{-\infty}^{+\infty} \mathrm{d}y K(x,y)\alpha(y), \tag{A.3.33a}$$

$$\beta'(x) = \int_{-\infty}^{+\infty} \mathrm{d}y K(x,y)\beta(y). \tag{A.3.33b}$$

From (A.3.16) and (A.3.30), we obtain

$$\mathcal{D}[K\varphi]\mathcal{D}[K\varphi^*] = (\det K)^2 \mathcal{D}[\varphi]\mathcal{D}\left[\varphi^*\right]. \tag{A.3.34}$$

**Application:**

(1)
$$\begin{aligned}\int \mathcal{D}[\alpha]F[\alpha]\exp\left[-\tfrac{1}{2}\alpha D\alpha + \mathrm{i}\phi\alpha\right] &= F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\phi}\right]\int \mathcal{D}[\alpha]\exp\left[-\tfrac{1}{2}\alpha D\alpha + \mathrm{i}\phi\alpha\right] \\ &= (\det D)^{-1/2} F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\phi}\right]\exp\left[-\tfrac{1}{2}\phi D^{-1}\phi\right],\end{aligned} \tag{A.3.35}$$

$$\int \mathcal{D}[\alpha]F[\alpha]\exp\left[-\tfrac{1}{2}\alpha D\alpha\right] = (\det D)^{-1/2} F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\phi}\right]\exp\left[-\tfrac{1}{2}\phi D^{-1}\phi\right]\big|_{\phi=0}\,. \tag{A.3.36}$$

(2)
$$\begin{aligned}\int &\mathcal{D}[\varphi]\mathcal{D}[\varphi^*]F[\varphi,\varphi^*]\exp[-\varphi^* D\varphi + \mathrm{i}(\chi^*\varphi + \varphi^*\chi)] \\ &= F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\chi^*},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\chi}\right]\int \mathcal{D}[\varphi]\mathcal{D}[\varphi^*]\exp[-\varphi^* D\varphi + \mathrm{i}(\chi^*\varphi + \varphi^*\chi)] \\ &= (\det D)^{-1} F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\chi^*},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\chi}\right]\exp\left[-\chi^* D^{-1}\chi\right],\end{aligned} \tag{A.3.37}$$

$$\begin{aligned}\int &\mathcal{D}[\varphi]\mathcal{D}[\varphi^*]F[\varphi,\varphi^*]\exp[-\varphi^* D\varphi] \\ &= (\det D)^{-1} F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\chi^*},\frac{1}{\mathrm{i}}\frac{\delta}{\delta\chi}\right]\exp[-\chi^* D^{-1}\chi]\,\big|_{\chi=\chi^*=0}\,.\end{aligned} \tag{A.3.38}$$

**Fermion Number Functional Integration:** We have also in the case of Grassman algebra

$$\int \mathcal{D}[\eta]\mathcal{D}[\eta^\dagger] \exp[-\eta^\dagger \eta] = 1, \tag{A.3.39}$$

$$\int \mathcal{D}[\eta]\mathcal{D}[\eta^\dagger] \exp[\mathrm{i}(\eta^\dagger \zeta + \zeta^\dagger \eta) - \eta^\dagger \eta] = \exp\left[-\zeta^\dagger \zeta\right]. \tag{A.3.40}$$

just like (A.3.17) and (A.3.20), where $\eta$, $\eta^\dagger$, $\zeta$ and $\zeta^\dagger$ are elements of the Grassman algebra. The consistency with (A.2.8) requires the change of function variable formula

$$\mathcal{D}[K\eta]\mathcal{D}[K\eta^\dagger] = (\det K)^{-2}\mathcal{D}[\eta]\mathcal{D}\left[\eta^\dagger\right]. \tag{A.3.41}$$

When we use the Dirac spinors, we define the Pauli adjoint $\bar{\eta}$ by

$$\bar{\eta} \equiv \eta^\dagger \gamma_0. \tag{A.3.42}$$

By the sequence of change of function variables indicated below

$$\eta \to (\gamma_0)^{1/2}\eta, \qquad \zeta \to (\gamma_0)^{1/2}\zeta, \tag{A.3.43a}$$

$$\bar{\eta} \to \bar{\eta}(\gamma_0)^{1/2}, \qquad \bar{\zeta} \to \bar{\zeta}(\gamma_0)^{1/2}, \tag{A.3.43b}$$

$$\det \gamma_0 = 1, \tag{A.3.44}$$

$$\eta^\dagger \eta = \bar{\eta}(\gamma_0)^{-1}\eta \to \bar{\eta}\eta, \qquad \zeta^\dagger \eta + \eta^\dagger \zeta \to \bar{\zeta}\eta + \bar{\eta}\zeta, \tag{A.3.45}$$

$$\mathcal{D}[\eta]\mathcal{D}[\eta^\dagger] \to \mathcal{D}[\eta]\mathcal{D}\left[\bar{\eta}\right], \tag{A.3.46}$$

we obtain from (A.3.39), (A.3.40) and (A.3.41) that

$$\int D[\eta]D[\bar{\eta}] \exp[-\bar{\eta}\eta + \mathrm{i}(\bar{\zeta}\eta + \bar{\eta}\zeta)] = \exp\left[-\bar{\zeta}\zeta\right]. \tag{A.3.47}$$

Corresponding to (A.3.21), we obtain

$$\int D[\eta]D[\bar{\eta}]F[\eta, \bar{\eta}] \exp[-\bar{\eta}\eta + \mathrm{i}(\bar{\zeta}\eta + \bar{\eta}\zeta)]$$

$$= F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\zeta}}, \mathrm{i}\frac{\delta}{\delta\zeta}\right] \exp\left[-\bar{\zeta}\zeta\right]. \tag{A.3.48}$$

We have the change of function variable formula for the Grassman number function as

$$\mathcal{D}[K\eta]\mathcal{D}[K\bar{\eta}] = (\det K)^{-2}\mathcal{D}[\eta]\mathcal{D}\left[\bar{\eta}\right]. \tag{A.3.49}$$

**Application:**

(3)
$$\begin{aligned}
&\int \mathcal{D}[\eta]\mathcal{D}[\bar{\eta}]F[\eta,\bar{\eta}]\exp[-\bar{\eta}D\eta + \mathrm{i}(\bar{\zeta}\eta + \bar{\eta}\zeta)] \\
&\quad = F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\zeta}}, \mathrm{i}\frac{\delta}{\delta\zeta}\right]\int \mathcal{D}[\eta]\mathcal{D}[\bar{\eta}]\exp[-\bar{\eta}D\eta + \mathrm{i}(\bar{\zeta}\eta + \bar{\eta}\zeta)] \\
&\quad = (\det D)F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\zeta}}, \mathrm{i}\frac{\delta}{\delta\zeta}\right]\exp\left[-\bar{\zeta}D^{-1}\zeta\right].
\end{aligned} \tag{A.3.50}$$

$$\int \mathcal{D}[\eta]\mathcal{D}[\bar{\eta}]F[\eta,\bar{\eta}]\exp[-\bar{\eta}D\eta] = (\det D)F\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\zeta}}, \mathrm{i}\frac{\delta}{\delta\zeta}\right]\exp[-\bar{\zeta}D^{-1}\zeta]\,|_{\zeta=\bar{\zeta}=0}\,. \tag{A.3.51}$$

**Functional Fourier Transform:** We define the functional Fourier transform $\tilde{F}[\bar{\zeta},\zeta]$ of the functional $F[\eta,\bar{\eta}]$ of the complex anticommuting fermion function $\eta(x)$ by

$$F[\eta,\bar{\eta}] = \int \mathcal{D}[\zeta]\mathcal{D}[\bar{\zeta}]\tilde{F}[\bar{\zeta},\zeta]\exp\left[\mathrm{i}(\bar{\zeta}\eta + \bar{\eta}\zeta)\right], \tag{A.3.52}$$

and its inverse by

$$\tilde{F}[\bar{\zeta},\zeta] = \int \mathcal{D}[\eta]\mathcal{D}[\bar{\eta}]F[\eta,\bar{\eta}]\exp\left[-\mathrm{i}(\bar{\zeta}\eta + \bar{\eta}\zeta)\right]. \tag{A.3.53}$$

# A.4 Gauge Invariance of $\mathcal{D}[A_{\alpha\mu}]\Delta_F[A_{\alpha\mu}]$

We introduce generic notation for the field. We let $\phi_a$ represent the generic field variable:

$$\phi_a = \begin{cases} \phi_i(x), & a=(i,x), \quad \text{scalar field}, \\ \psi_n(x), & a=(n,x), \quad \text{spinor field}, \\ A_{\alpha\mu}(x), & a=(\alpha,\mu,x), \text{ gauge field}. \end{cases} \tag{A.4.1a, b, c}$$

We define the infinitesimal gauge transform $\phi_a^g$ of $\phi_a$ by

$$\phi_a^g = \phi_a + (\mathrm{i}\Gamma_{ab}^{\gamma}\phi_b + \Lambda_a^{\gamma})\varepsilon_\gamma + O(\varepsilon^2), \tag{A.4.2a}$$

where

$$\varepsilon_\gamma = \varepsilon_\gamma(x^{(\gamma)}), \qquad \text{infinitesimal function}. \tag{A.4.2b}$$

We define $\Gamma_{ab}^{\gamma}$ and $\Lambda_a^{\gamma}$ by

**Scalar Field:**

$$a=(i,x^{(i)}),\ b=(j,x^{(j)}),$$

$$\Gamma_{ab}^{\gamma} = (\theta_\gamma)_{ij}\delta^4(x^{(i)}-x^{(j)})\delta^4(x^{(i)}-x^{(\gamma)}),$$

$$\Lambda_a^{\gamma} = 0, \tag{A.4.3a}$$

**Spinor Field:**

$$a=(n,x^{(n)}),\ b=(m,x^{(m)}),$$

$$\Gamma_{ab}^{\gamma} = (t_\gamma)_{nm}\delta^4(x^{(n)}-x^{(m)})\delta^4(x^{(n)}-x^{(\gamma)}),$$

$$\Lambda_a^{\gamma} = 0, \tag{A.4.3b}$$

**Gauge Field:**

$$a=(\alpha\mu,x^{(\alpha)}),\ b=(\beta\nu,x^{(\beta)}),$$

$$\Gamma_{ab}^{\gamma}=(\theta_{\gamma}^{\mathrm{adj}})_{\alpha\beta}\delta_{\mu\nu}\delta^4(x^{(\alpha)}-x^{(\beta)})\delta^4(x^{(\alpha)}-x^{(\gamma)}),$$

$$\Lambda_{a}^{\gamma}=-\delta_{\alpha\gamma}\partial_{\mu}^{(\alpha)}\delta^4(x^{(\alpha)}-x^{(\gamma)}). \tag{A.4.3c}$$

We use the standard convention that repeated indices are summed over and integrated over for discrete indices and continuous indices, respectively. We have the group property of the gauge transformation (A.4.2a) as

$$\Gamma^{\alpha}(\mathrm{i}\Gamma\phi+\Lambda)^{\beta}-\Gamma^{\beta}(\mathrm{i}\Gamma\phi+\Lambda)^{\alpha}=\mathrm{i}C_{\alpha\beta\gamma}(\mathrm{i}\Gamma\phi+\Lambda)^{\gamma}. \tag{A.4.4}$$

As the gauge-fixing condition, we employ

$$F_{\alpha}(\phi_a)=a_{\alpha}(x^{(\alpha)}),\qquad \alpha=1,\ldots,N. \tag{A.4.5}$$

We have the Faddeev–Popov determinant as

$$\Delta_F[\phi_a]\equiv \mathrm{Det}M_F(\phi_a)=\exp\left[\mathrm{Tr}\ln M_F(\phi_a)\right], \tag{A.4.6a}$$

$$M_{F;\alpha x,\beta y}(\phi_a)\equiv\frac{\delta F_{\alpha}(\phi_a^g(x))}{\delta\varepsilon_{\beta}(y)}\Big|_{g=1}=\frac{\delta F_{\alpha}(\phi_a)}{\delta\phi_a}(\mathrm{i}\Gamma_{ab}^{\beta}\phi_b+\Lambda_a^{\beta}), \tag{A.4.6b}$$

$$F_{\alpha}(\phi_a^g)=F_{\alpha}(\phi_a)+M_{F;\alpha\beta}(\phi_a)\varepsilon_{\beta}+O(\varepsilon^2). \tag{A.4.6c}$$

Under the nonlinear gauge transformation $g_0$,

$$\varepsilon_{\alpha}^{g_0}(x)=(M_F^{-1}(\phi_a))_{\alpha\beta}\lambda_{\beta}(x), \tag{A.4.7}$$

where $\lambda_{\beta}(x)$ is an infinitesimal function independent of $\phi_a$, the Faddeev–Popov determinant gets transformed into

$$\begin{aligned}\delta^{g_0}(\ln\Delta_F[\phi_a]) &= \delta^{g_0}(\mathrm{Tr}\ln M_F(\phi_a))\\ &= \mathrm{Tr}(M_F^{-1}(\phi_a)\delta^{g_0}M_F(\phi_a))\\ &= \mathrm{Tr}\left\{M_F^{-1}(\phi_a)\frac{\delta M_F(\phi_a)}{\delta\phi_b}\right\}\delta^{g_0}\phi_b\\ &= (M_F^{-1}(\phi_a))_{\gamma\varepsilon}\frac{\delta M_F(\phi_a)_{\varepsilon\gamma}}{\delta\phi_b}\\ &\qquad\times(\mathrm{i}\Gamma\phi+\Lambda)_b^{\alpha}(M_F^{-1}(\phi_a))_{\alpha\beta}\lambda_{\beta}(x).\end{aligned} \tag{A.4.8}$$

The Jacobian of the nonlinear gauge transformation (A.4.7) is given by

$$
\begin{aligned}
\ln \mathcal{J}\left(\left.\frac{\partial \phi_a^{g_0}}{\partial \phi_b}\right|_{g_0=1}\right) &= \operatorname{Tr}\ln\left\{1 + \mathrm{i}\Gamma^{\gamma}_{ab}\varepsilon^{g_0}_{\gamma} + (\mathrm{i}\Gamma\phi + \Lambda)^{\gamma}_{a}\frac{\delta\varepsilon^{g_0}_{\gamma}}{\delta\phi_b}\right\}\Bigg|_{g_0=1} \\
&= \operatorname{Tr}\Bigg\{\mathrm{i}\Gamma^{\gamma}_{ab}\varepsilon^{g_0}_{\gamma} - (\mathrm{i}\Gamma\phi + \Lambda)^{\gamma}_{a}(M_F^{-1}(\phi_a))_{\gamma\varepsilon} \\
&\qquad \times \frac{\delta M_F(\phi_a)_{\varepsilon\alpha}}{\delta\phi_b}(M_F^{-1}(\phi_a))_{\alpha\beta}\lambda_\beta(x)\Bigg\} \\
&= -(\mathrm{i}\Gamma\phi + \Lambda)^{\gamma}_{a}(M_F^{-1}(\phi_a))_{\gamma\varepsilon}\frac{\delta M_F(\phi_a)_{\varepsilon\alpha}}{\delta\phi_a} \\
&\qquad \times (M_F^{-1}(\phi_a))_{\alpha\beta}\lambda_\beta(x),
\end{aligned} \tag{A.4.9}
$$

where

$$\operatorname{Tr}\Gamma^{\gamma}_{ab} = 0$$

is used in (A.4.9). From (A.4.8) and (A.4.9), we obtain

$$
\begin{aligned}
&\delta^{g_0}(\ln \Delta_F[\phi_a]) + \ln \mathcal{J}\left(\left.\frac{\partial \phi_a^{g_0}}{\partial \phi_b}\right|_{g_0=1}\right) \\
&\quad = (M_F^{-1}(\phi_a))_{\gamma\varepsilon}(M_F^{-1}(\phi_a))_{\alpha\beta}\lambda_\beta(x) \\
&\qquad \times \left\{\frac{\delta M_F(\phi_a)_{\varepsilon\gamma}}{\delta\phi_b}(\mathrm{i}\Gamma\phi + \Lambda)^{\alpha}_{b} - \frac{\delta M_F(\phi_a)_{\varepsilon\alpha}}{\delta\phi_b}(\mathrm{i}\Gamma\phi + \Lambda)^{\gamma}_{b}\right\}.
\end{aligned} \tag{A.4.10}
$$

From (A.4.6), we have for the $\{\cdots\}$ part:

$$
\begin{aligned}
\{\cdots\} &= \frac{\delta^2 F_\varepsilon(\phi)}{\delta\phi_a\delta\phi_b}(\mathrm{i}\Gamma\phi + \Lambda)^{\gamma}_{a}(\mathrm{i}\Gamma\phi + \Lambda)^{\alpha}_{b} + \frac{\delta F_\varepsilon(\phi)}{\delta\phi_a}(\mathrm{i}\Gamma^{\gamma}_{ab})(\mathrm{i}\Gamma\phi + \Lambda)^{\alpha}_{b} \\
&\quad -(\alpha \leftrightarrow \gamma) \\
&= \mathrm{i}\frac{\delta F_\varepsilon(\phi)}{\delta\phi_a}\{\Gamma^{\gamma}(\mathrm{i}\Gamma\phi + \Lambda)^{\alpha} - \Gamma^{\alpha}(\mathrm{i}\Gamma\phi + \Lambda)^{\gamma}\}_a \\
&= -C_{\alpha\beta\gamma}\frac{\delta F_\varepsilon(\phi)}{\delta\phi_a}(\mathrm{i}\Gamma\phi + \Lambda)^{\beta}_{a} \\
&= -C_{\alpha\beta\gamma}M_F(\phi_a)_{\varepsilon\beta}.
\end{aligned} \tag{A.4.11}
$$

Hence, from (A.4.10) and (A.4.11), we obtain

$$
\begin{aligned}
&\delta^{g_0}(\ln \Delta_F[\phi_a]) + \ln \mathcal{J}\left(\left.\frac{\partial \phi_a^{g_0}}{\partial \phi_b}\right|_{g_0=1}\right) \\
&= \varepsilon^{g_0}_{\alpha}(x)(-C_{\alpha\beta\gamma})(M_F^{-1}(\phi_a)M_F(\phi_a))_{\gamma\beta} \\
&= \varepsilon^{g_0}_{\alpha}(x)(-C_{\alpha\beta\gamma})\delta_{\gamma\beta} = 0.
\end{aligned} \tag{A.4.12}
$$

Since we know

$$\delta^{g_0}(\ln \mathcal{D}[\phi_a]) = \ln \mathcal{J}\left(\left.\frac{\partial \phi_a^{g_0}}{\partial \phi_b}\right|_{g_0=1}\right), \tag{A.4.13}$$

we obtain

$$\delta^{g_0}(\ln \Delta_F[\phi_a]) + \delta^{g_0}(\ln \mathcal{D}[\phi_a]) = 0,$$

or

$$\delta^{g_0}(\ln(\mathcal{D}[\phi_a]\Delta_F[\phi_a])) = 0. \tag{A.4.14}$$

Hence, we have

$$\mathcal{D}[\phi_a]\Delta_F[\phi_a] = \text{gauge invariant functional integration measure under } g_0. \tag{A.4.15}$$

(A.4.15) is used in Chap. 3.

# A.5 Minkowskian and Euclidean Spinors

We let the spinors in Minkowskian space-time and Euclidean space-time be $\psi_{\mathrm{M}}$ and $\psi_{\mathrm{E}}$. We represent the $\gamma$ matrices in Minkowskian space-time as $\{\gamma_\mu\}_{\mu=0}^{3}$and in Euclidean space-time as $\{\gamma_k\}_{k=1}^{4}$. Corresponding to the analytic continuation

$$t = -\mathrm{i}\tau, \tag{A.5.1}$$

we define $\gamma^4$ by

$$\gamma^0 \equiv -\gamma^4. \tag{A.5.2}$$

We have the following anticommutators of the $\gamma$ matrices.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, 3, \tag{A.5.3M}$$

$$\{\gamma^k, \gamma^l\} = -2\delta^{kl}, \qquad k, l = 1, 2, 3, 4. \tag{A.5.3.E}$$

We define the Pauli adjoints, $\bar{\psi}_{\mathrm{M}}$ and $\bar{\psi}_{\mathrm{E}}$, by

$$\bar{\psi}_{\mathrm{M}} \equiv \psi_{\mathrm{M}}^\dagger \gamma^0, \qquad \bar{\psi}_{\mathrm{E}} \equiv \psi_{\mathrm{E}}^\dagger \gamma^4. \tag{A.5.4}$$

We have the following correspondence:

$$\bar{\psi}_{\mathrm{M}} \leftrightarrow -\mathrm{i}\bar{\psi}_{\mathrm{E}}, \tag{A.5.5a}$$

$$\psi_{\mathrm{M}} \leftrightarrow \mathrm{i}\psi_{\mathrm{E}}, \tag{A.5.5b}$$

$$\bar{\psi}_{\mathrm{M}}\psi_{\mathrm{M}} \leftrightarrow \bar{\psi}_{\mathrm{E}}\psi_{\mathrm{E}}, \tag{A.5.5c}$$

$$\bar{\psi}_{\mathrm{M}} \left\{ \gamma^0 \left( \mathrm{i}\frac{\partial}{\partial t} + \mu \right) + \gamma^k \mathrm{i}\partial_k \right\} \psi_{\mathrm{M}} \leftrightarrow \bar{\psi}_{\mathrm{E}} \left\{ \mathrm{i}\gamma^k \partial_k + \mathrm{i}\gamma^4 \left( \frac{\partial}{\partial \tau} - \mu \right) \right\} \psi_{\mathrm{E}}. \tag{A.5.5d}$$

These correspondences are used in Sect. 4.2.1 of Chap. 4.

## A.6 Multivariate Normal Analysis

We let

$$\boldsymbol{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$$

be an $n$-dimensional vector random variable. We define the mean vector $\boldsymbol{\mu}$ by

$$\boldsymbol{\mu} = E\boldsymbol{X}. \tag{A.6.1}$$

We define the covariance $\boldsymbol{\Sigma}_{XY}$ of vector random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ by

$$\boldsymbol{\Sigma}_{XY} = E\{(\boldsymbol{X} - \boldsymbol{\mu}_{\boldsymbol{X}})(\boldsymbol{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})^{\mathrm{T}}\}, \tag{A.6.2}$$

whose $(i,j)$-element is given by

$$\boldsymbol{\sigma}_{X_i Y_j} = E\{(X_i - \mu_{X_i})(Y_j - \mu_{Y_j})\}.$$

$\boldsymbol{\Sigma}_{XX}$ is the covariance matrix of the vector random variable $\boldsymbol{X}$. It is non-negative definite and real symmetric matrix. We define the moment generating function $M_{\boldsymbol{X}}(\boldsymbol{t})$ of the vector random variable $\boldsymbol{X}$ by

$$M_{\boldsymbol{X}}(\boldsymbol{t}) \equiv E\{\exp[\boldsymbol{t}^{\mathrm{T}}\boldsymbol{X}]\} = E\left\{\exp\left[\sum_{k=1}^{n} t_k X_k\right]\right\}. \tag{A.6.3}$$

Under the linear transformation ($\boldsymbol{Y}(m \times 1) \to \boldsymbol{X}(n \times 1)$),

$$\boldsymbol{X} = \boldsymbol{a} + B\boldsymbol{Y}, \tag{A.6.4a}$$

$$\boldsymbol{a} = n \times 1 \text{ vector}, \qquad B = n \times m \text{ matrix}, \tag{A.6.4b}$$

we have

$$\boldsymbol{\mu}_{\boldsymbol{X}} = \boldsymbol{a} + B\boldsymbol{\mu}_{\boldsymbol{Y}}, \tag{A.6.5a}$$

$$\boldsymbol{\Sigma}_{XX} = B\boldsymbol{\Sigma}_{YY}B^{T}. \tag{A.6.5b}$$

We let $\boldsymbol{Y} = (Y_1, \ldots, Y_m)^{\mathrm{T}}$ be the vector random variable whose components $Y_j'$s are the independent identically distributed $\mathcal{N}(0,1)$ random variables. We have the probability density function $f_{\boldsymbol{Y}}(\boldsymbol{y})$ of $\boldsymbol{Y}$ given by

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \frac{1}{(2\pi)^{m/2}} \exp\left[-\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}\right] = \frac{1}{(2\pi)^{m/2}} \exp\left[-\frac{1}{2}\sum_{j=1}^{m} y_j^2\right]. \tag{A.6.6}$$

We call the vector random variable $\boldsymbol{X}$ which is given by the linear combination of $\boldsymbol{Y}$

$$\boldsymbol{X} = \boldsymbol{a} + B\boldsymbol{Y} \tag{A.6.7a}$$

$$Y_j = \text{iid. } \mathcal{N}(0,1), \qquad j = 1, \ldots, m, \tag{A.6.7b}$$

a multivariate normal random variable and

$$\boldsymbol{\mu}_{\boldsymbol{X}} = \boldsymbol{a}, \qquad \boldsymbol{\Sigma}_{XX} = BB^{\mathrm{T}}. \tag{A.6.8}$$

We write the law of the distribution of $\boldsymbol{X}$ as

$$\mathcal{L}(\boldsymbol{X}) = \mathcal{N}_n(\boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{XX}).$$

The moment generating function $M_{\boldsymbol{X}}(\boldsymbol{t})$ of the multivariate normal $\boldsymbol{X}$ is given by

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = \exp\left[\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\mu}_{\boldsymbol{X}} + \frac{1}{2}\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\Sigma}_{XX}\boldsymbol{t}\right]. \tag{A.6.9}$$

Hence, the law of the distribution of the multivariate normal $\boldsymbol{X}$ is uniquely determined by $\boldsymbol{\mu}_{\boldsymbol{X}}$ and $\boldsymbol{\Sigma}_{XX}$. Furthermore, given a nonnegative definite real symmetric matrix $\boldsymbol{\Sigma}$, there exists a (not necessarily unique) square matrix $B$ such that

$$\boldsymbol{\Sigma} = BB^{\mathrm{T}}. \tag{A.6.10}$$

We observe that an arbitrary $\boldsymbol{\mu}$ and an arbitrary nonnegative definite real symmetric matrix $\boldsymbol{\Sigma}$ determine the multivariate normal distribution $\mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. When $\boldsymbol{\Sigma}_{XX}$ is positive definite, we have the probability density function $f_{\boldsymbol{X}}(\boldsymbol{x})$ of $\boldsymbol{X}$ given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(\det \boldsymbol{\Sigma}_{XX})^{1/2}} \exp\left[-\tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{X}})^{\mathrm{T}}\boldsymbol{\Sigma}_{XX}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_{\boldsymbol{X}})\right]. \tag{A.6.11}$$

Next, we consider the case when $\boldsymbol{\Sigma}_{XX}$ is singular (or $B$ is singular). We let the rank of $\boldsymbol{\Sigma}_{XX}$ be $n^* < n$. The $n^*$ components of the vector random variable $\boldsymbol{X}$ are represented by the $n^* \times n^*$ nonsingular matrix $B$ and the $n^*$ $Y_j \sim$ iid. $\mathcal{N}(0,1)$ random variables, while the remaining $(n-n^*)$ components of $\boldsymbol{X}$ are given by a linear combination of the $n^*$ independent components of $\boldsymbol{X}$. For a comparison with the singular Gaussian integral discussed in Appendix 1, we note the correspondence

$$n \longleftrightarrow f, \tag{A.6.12a}$$

$$n^* \longleftrightarrow f - p, \tag{A.6.12b}$$

$$n - n^* \longleftrightarrow p. \tag{A.6.12c}$$

Application to the path integral quantization of non-Abelian gauge field theory is discussed in Appendix 1.

## A.7 Bibliography


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**A.2**

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**A.5**

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**A.6**

(**T-65**) Morrison, D.F.; ″Multivariate Statistical Methods″, McGraw-Hill, 1976, New York, Chapter 3, Sections 3.3 and 3.4.


In Chaps. 1 through 4, we followed the standard folklore in writing down the path integral formula: a piecewise continuous but nowhere differentiable **quantum path** was interpolated by a continuous and differentiable function. To remedy this deficiency, we use **Hida calculus** invented for the generalized theory of stochastic processes with values in the distribution. The bosonic path integral formula follows with a three line proof by using Hida Calculus, while the fermionic path integral formula must await further progress in mathematics.

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